Introduzione alle masse e ai mescolamenti dei neutrini (I)

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The lectures are intended for a broad audience of students or researchers from different fields in particle physics

The goal is to "get you interested" in neutrino physics, by recalling basic neutrino properties and phenomena, which will be further discussed in more specialized lectures

Some simple exercises are also proposed (with solutions)

People interested in further reading can usefully browse the "Neutrino Unbound" website: www.nu.to.infn.it , or just mail me for advice about specific topics: eligio.lisi@ba.infn.it

Feel free to stop me and ask questions at any time!

Outline:

Pedagogical Introduction
Neutrino masses and spinor fields
Neutrinoless double beta decay
2ν & 3ν oscillations in vacuum
[Homework]

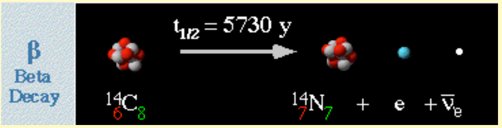
I

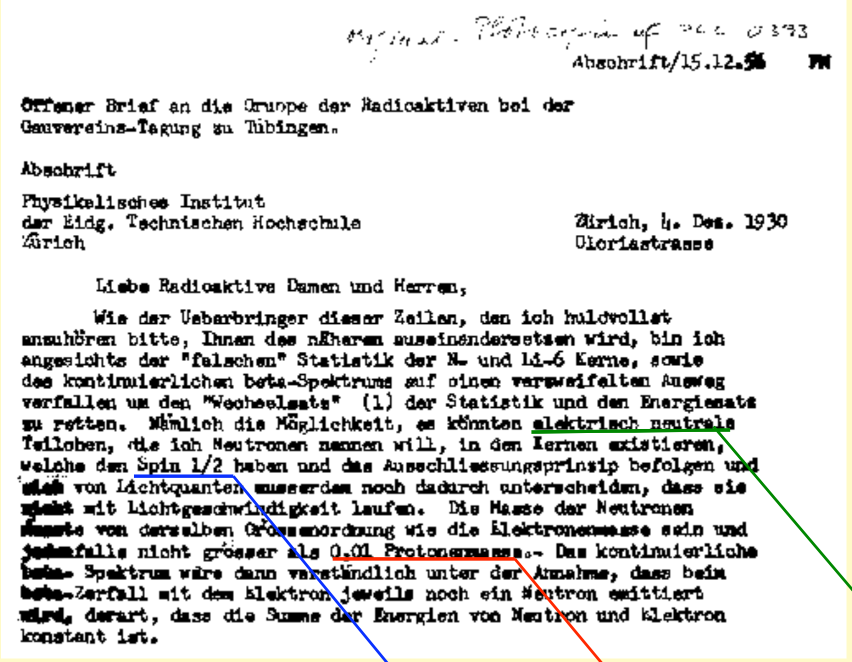
Recap
2ν oscillations in matter
Solar and KamLAND oscillations
Absolute neutrino masses
[Homework]

II

2010: the 80th Neutrino Birthday!

The neutrino was invented in 1930 by Wolfgang Pauli as a "desperate remedy" to explain the continuous β-ray spectrum via a 3-body decay, e.g.,





Abschrift/15.12.56 PM

Offener Brief an die Gruppe der Radioaktiven bei der Gauvereins-Tagung zu Tübingen.

Abschrift

Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Dez. 1930
Gloriastrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich huldvollst anzuhören bitte, Ihnen des näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweifelten Ausweg verfallen um den "Wechselsatz" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich von Lichtquanten ausserdem noch dadurch unterscheiden, dass sie nicht mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grossenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0,01 Protonenmasse.- Das kontinuierliche beta- Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, derart, dass die Summe der Energien von Neutron und Elektron konstant ist.

Kinematics: spin 1/2, tiny mass, zero electric harge

The name “neutrino” (=“little neutral one”, in Italian) was actually invented by Enrico Fermi, who first proposed in 1933-34 a theory for its **dynamics** (weak interactions)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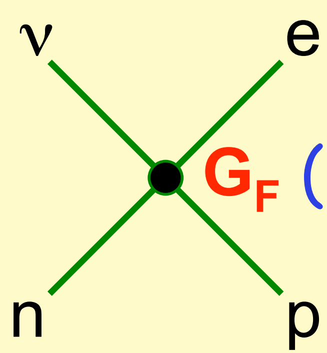
LA RICERCA SCIENTIFICA

ED IL PROGRESSO TECNICO NELL'ECONOMIA NAZIONALE

Tentativo di una teoria dell'emissione dei raggi “beta”

Nota del prof. ENRICO FERMI

Riassunto: Teoria della emissione dei raggi β delle sostanze radioattive, fondata sull'ipotesi che gli elettroni emessi dai nuclei non esistano prima della disintegrazione ma vengano formati, insieme ad un neutrino, in modo analogo alla formazione di un quanto di luce che accompagna un salto quantico di un atomo. Confronto della teoria con l'esperienza.

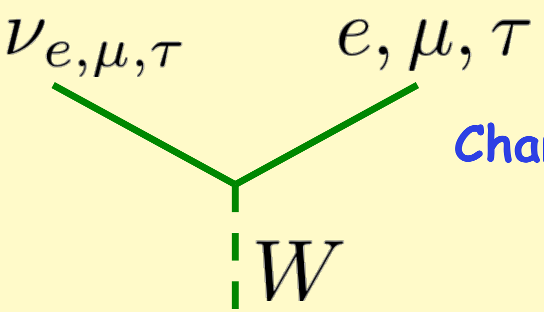


$\mathbf{G_F}$ (Fermi constant)

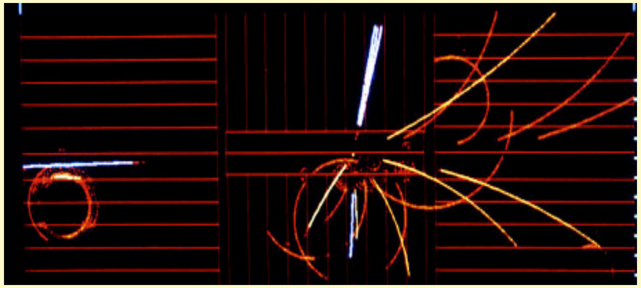
Many decades of research have revealed other properties of the neutrino. For instance, there are 3 different neutrino "flavors"

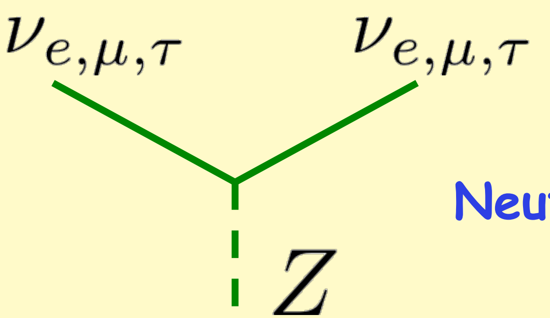
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \begin{array}{l} \leftarrow \quad q = 0 \\ \leftarrow \quad q = -1 \end{array} \quad (\Delta q = 1)$$

and their Fermi interactions are mediated by a charged vector boson W, with a neutral counterpart, the Z boson

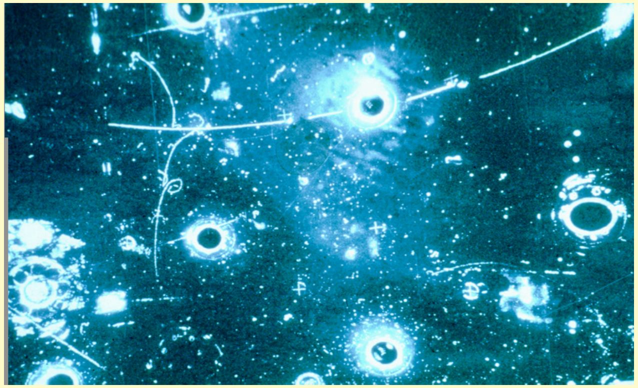


Charged Current (Δq=1)





Neutral Current (Δq=0)

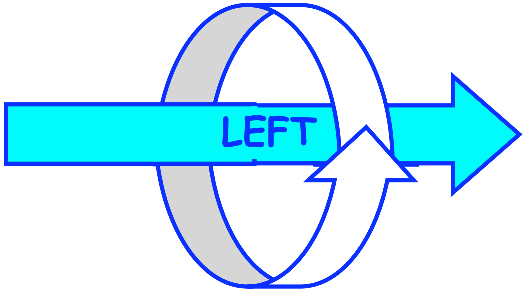


(For the other vertex in W, Z exchange: See lecture by L. Ludovici)

Such interactions are chiral (= not mirror-symmetric):

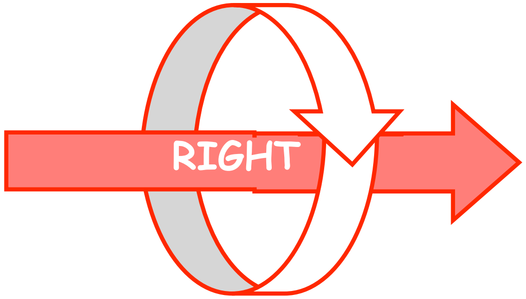
Neutrinos are created in
a left-handed (LH) state

ν



Anti-nus are created in
a right-handed (RH) state

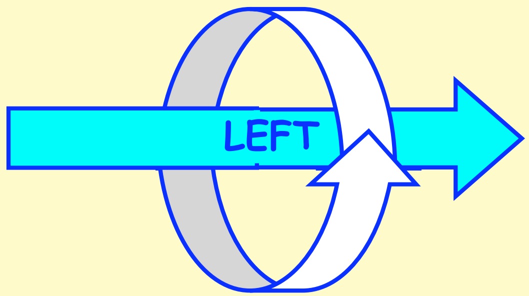
$\bar{\nu}$



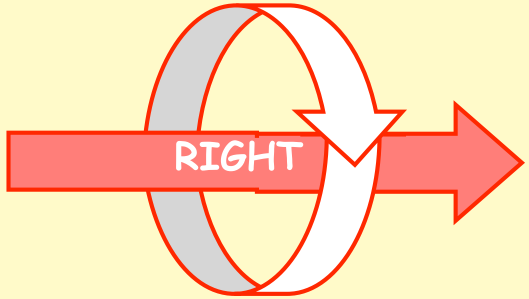
Neutrinos couldn't see themselves in a mirror… like vampires!

For massless neutrinos: handedness is a constant of motion

ν:

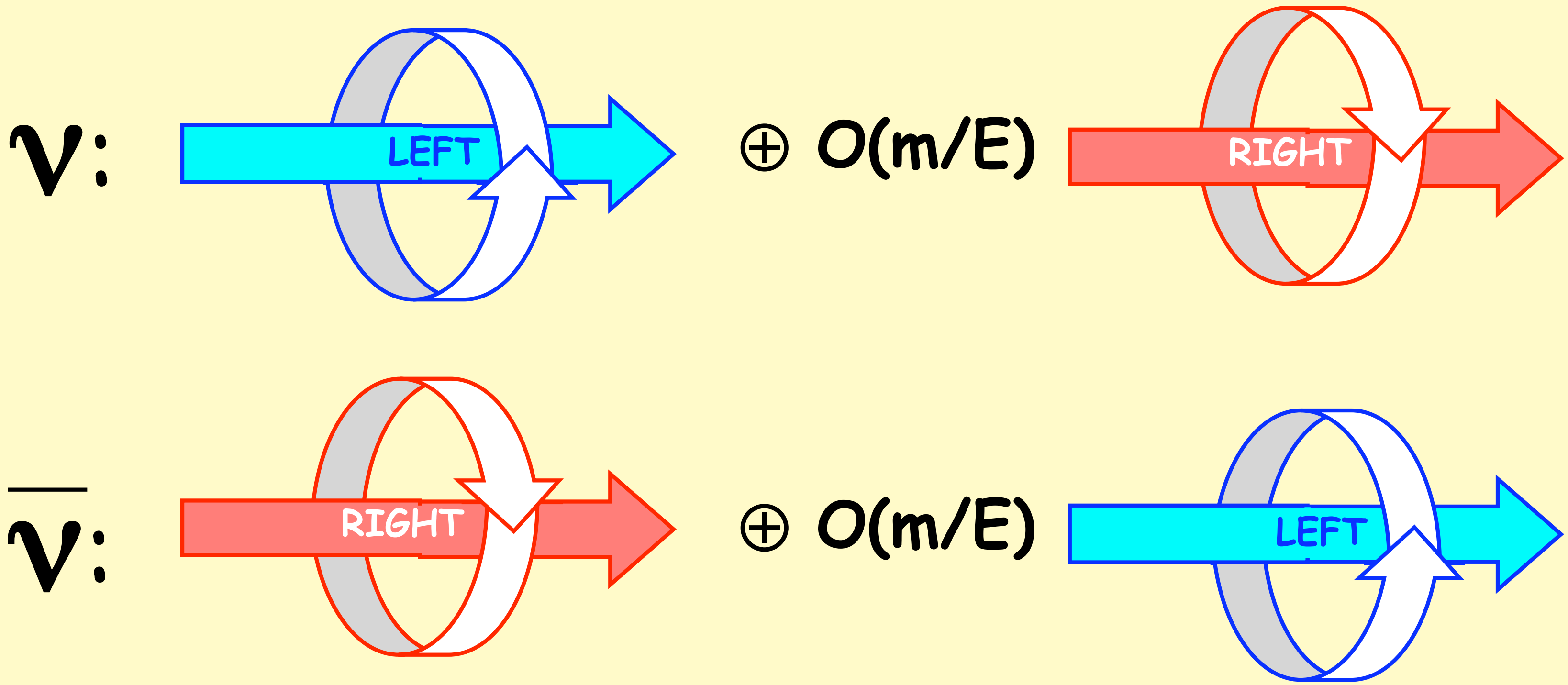


$\bar{\nu}$:



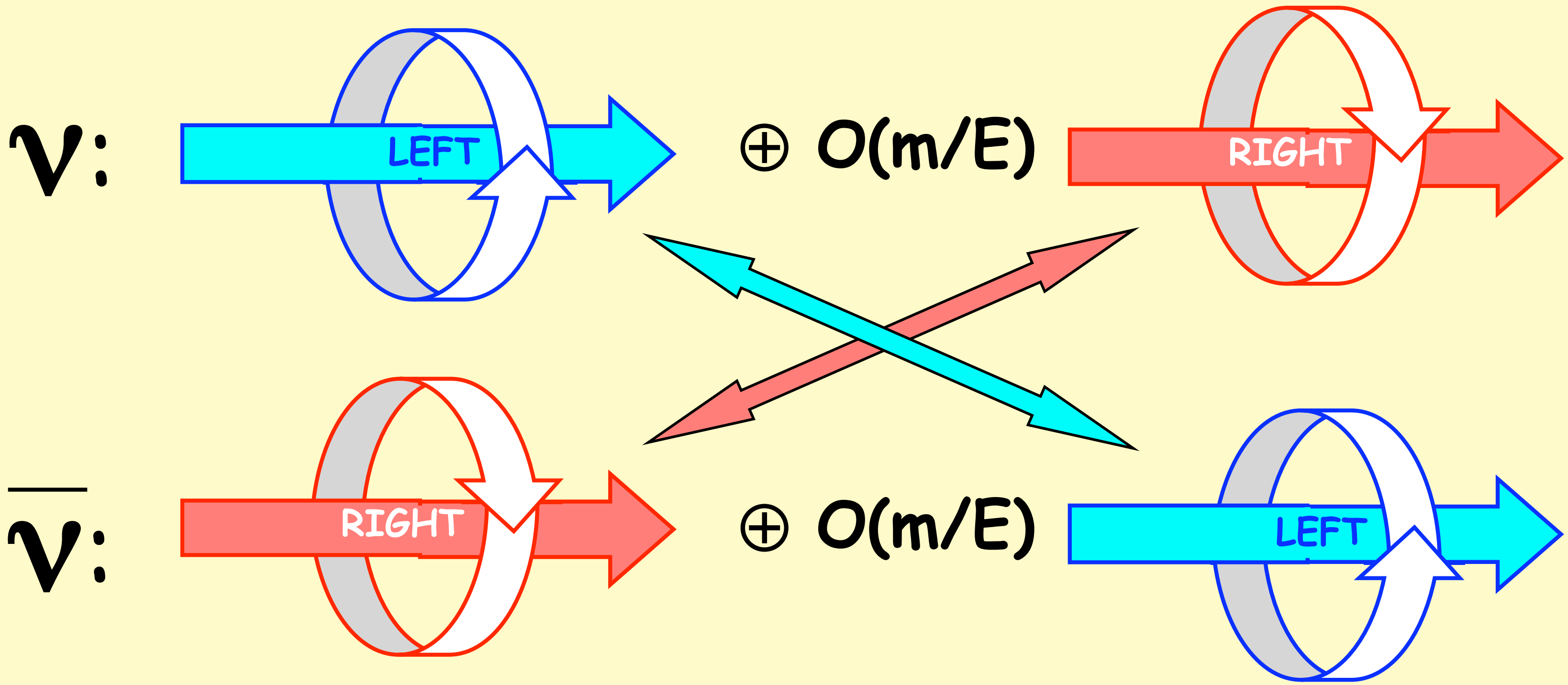
2 independent d.o.f.: massless ("Weyl") 2-spinor

But: massive ν can develop the "wrong" handedness at O(m/E)
(the Dirac equation mixes RH and LH states for $m_\nu \neq 0$):



If these 4 d.o.f. are independent: massive ("Dirac") 4-spinor
[→ Distinction between neutrinos and antineutrinos, as for electrically charged fermions. Can define a "lepton number"]

But, for neutral fermions, 2 components might be identical !



Massive ("Majorana") 4-spinor with 2 independent d.o.f.

[No distinction between neutrinos and antineutrinos, up to a phase:
A *very* neutral particle: no electric charge, no leptonic number…]

Exercise 1. Define the electron neutrino as the neutral particle emitted in β+ decay, and the electron antineutrino as the neutral particle emitted in β- decay. Reactions which have been observed:

$$\nu_e + n \to p + e^-$$

$$\overline{\nu}_e + p \to n + e^+$$

while the following reactions have not been observed:

$$\overline{\nu}_e + n \to p + e^-$$

$$\nu_e + p \to n + e^+$$

If neutrinos and antineutrinos are different (Dirac case), that's easy to understand. Try to understand the same (non)observations in the case of Majorana neutrinos.

Summary of options for neutrino spinor field:

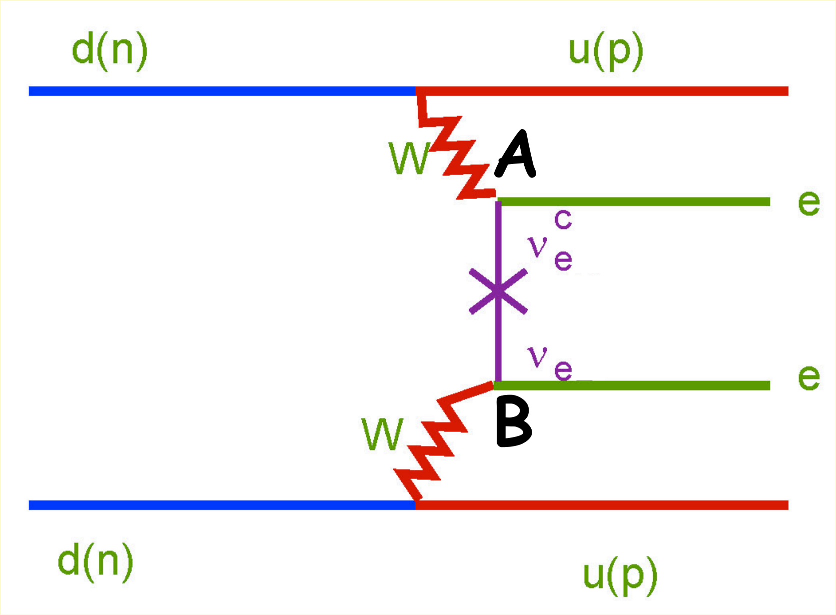
m=0, Weyl:	$\psi = \psi_R$ or $\psi = \psi_L$	massless field with 2 d.o.f.
m≠0, Majorana:	$\psi = \psi_R + \psi_R^c = \psi^c$ or $\psi = \psi_L + \psi_L^c = \psi^c$	massive field with 2 d.o.f.
m≠0, Dirac:	$\psi = \psi_R + \psi_L \neq \psi_c$	massive field with 4 d.o.f.

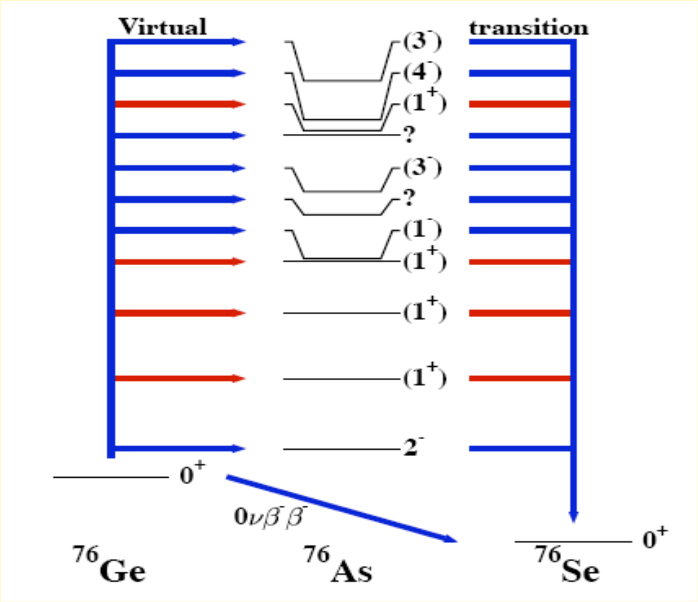
Conjugation operator: $\psi^c = \mathcal{C}(\psi) = i\gamma^2\psi^*$, $\psi_{\text{antiparticle}} = \mathcal{C}(\psi_{\text{particle}})$

Appendix: Majorana masses and "see-saw" mechanism [+ see talk by F. Feruglio]

Experiments: A unique experimental handle →

Neutrinoless double beta decay: $(A,Z) \rightarrow (A,Z+2)+2e$



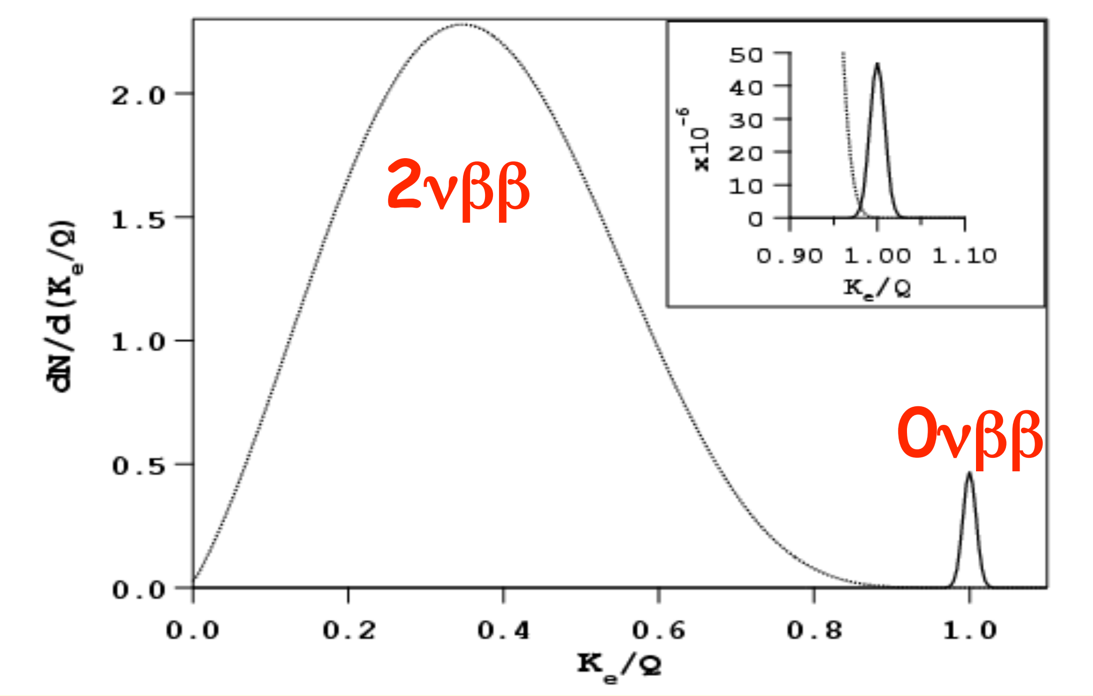


Can occur only for Majorana neutrinos. Intuitive picture:

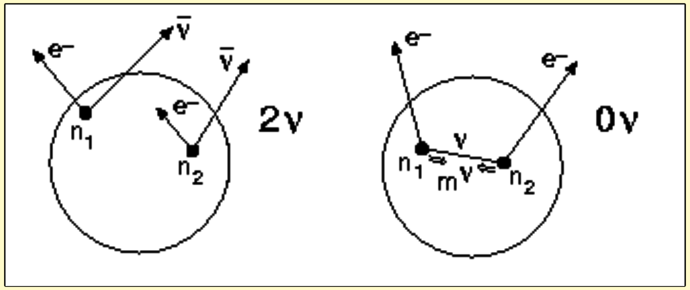
1) A RH antineutrino is emitted at point "**A**" together with an electron
2) If it is massive, at O(m/E) it develops a LH component (not possible if Weyl)
3) If neutrino=antineutrino, this component is a LH neutrino (not possible if Dirac)
4) The LH (Majorana) neutrino is absorbed at "**B**" where a 2nd electron is emitted

[EW part is "simple". Nuclear physics part is rather complicated and uncertain.]

Experimentally: Look at sum energy of both electrons



Need to see the 0νββ line emerge above background, at the endpoint of spectrum from "conventional" (and observed) 2νββ decay.



Very rare to detect (if it occurs): doubly-weak and suppressed by m/E. Need to be tenacious… like E. Fiorini (see next lecture)

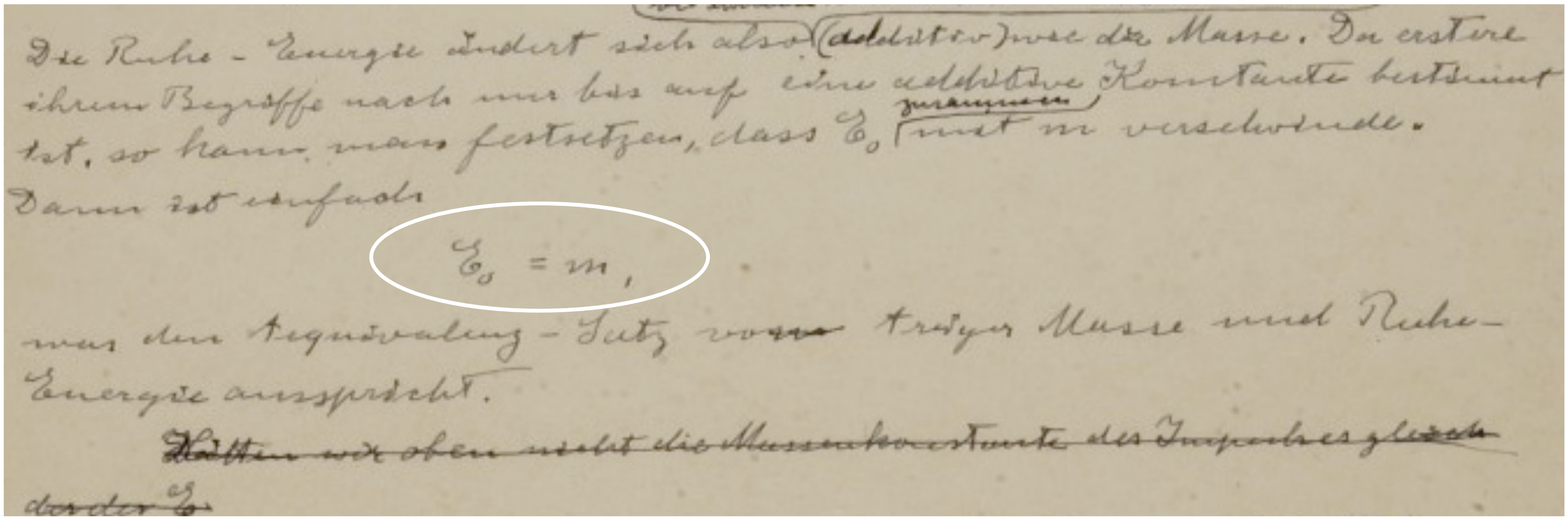
Recap: if neutrinos have mass, they can develop the **"wrong handedness"** with **amplitude of O(m_{ass}/E_{nergy}).** The only known chance to observe this tiny effect is **0νββ decay**.

But, if neutrinos are not only massive but mixed, they can also develop in the **"wrong flavor"** as a major consequence (**"neutrino flavor oscillations"**). This effect, despite being only of **O(m^2/E) in the phase**, can become observable over macroscopic distances (similar to optical interferometry).

We shall now discuss the phenomenon of flavor oscillations, going from simplified approximation to more realistic scenarios.

Neutrino flavor oscillations in vacuum (2ν)

The starting point is a century-old equation …



Die Ruhe-Energie ändert sich also (additiv) wie die Masse. Da erstere ihrem Begriffe nach nur bis auf eine additive Konstante bestimmt ist, so kann man festsetzen, dass $\mathcal{E}_0$ zusammen mit m verschwinde. Dann ist einfach

$$\mathcal{E}_0 = m,$$

was den Äquivalenz-Satz von träger Masse und Ruhe-Energie ausspricht.

~~Hätten wir oben nicht die Massenkonstante des Impulses gleich oder $\mathcal{E}_0$~~

… namely, for p≠0:

$$E = \sqrt{m^2 + p^2}$$

(in natural units)

Our ordinary experience takes place in the limit: $p \ll m$

$$E \simeq m + \frac{p^2}{2m}$$

... while for neutrinos the proper limit is: $p \gg m$

$$E \simeq p + \frac{m^2}{2p}$$

Energy difference between two neutrinos ν_i e ν_j with mass m_i e m_j in the same beam $(p_i = p_j \simeq E)$:

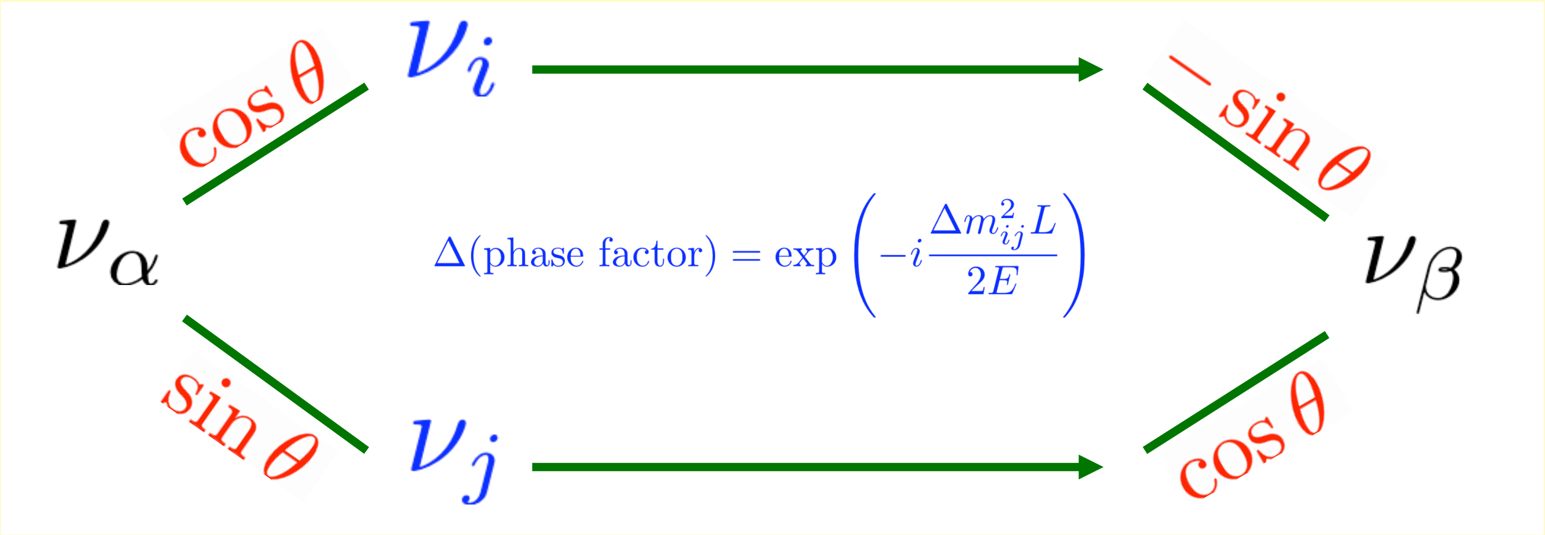
$$\Delta E \simeq \frac{\Delta m_{ij}^2}{2E}$$

PMNS*: neutrinos with definite mass (ν_i and ν_j) might have NO definite flavor (ν_α e ν_β), e.g.,

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_i \\ \nu_j \end{pmatrix}$$

***Pontecorvo; Maki, Nakagawa & Sakata**

Analogy with a two-slit interference experiment in vacuum:



This is the simplest case (only 2 neutrinos involved, no interactions with matter). It shows that, if neutrinos are massive and mixed (like quarks), then flavor is not a good quantum number during propagation. Indeed, it changes ("oscillates") significantly over a distance L (≈Δt) dictated by the uncertainty relation:

$$1 \sim \Delta E \Delta t \simeq \frac{m_i^2 - m_j^2}{2E} L$$

Exercise 2. Prove that a neutrino created with flavor α can develop a different flavor β with a periodical oscillation probability in **L/E**:

$$P(\nu_\alpha \to \nu_\beta) = 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right)$$

(B. Pontecorvo)

Amplitude
(vanishes for θ=0 or π/2)

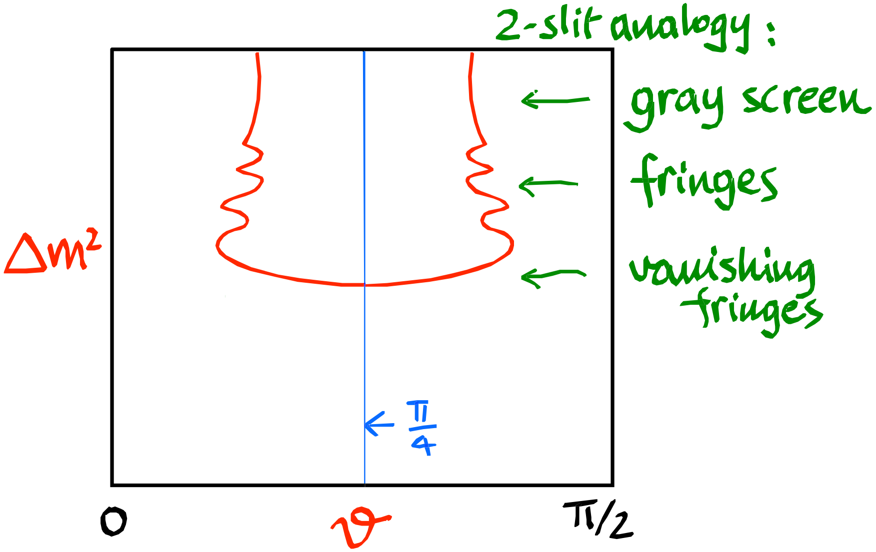
Phase difference
(vanishes for degenerate masses)

Note : This is the flavor "appearance" probability.
The flavor "disappearance" probability is the complement to 1.

Exercise 3. The oscillation effect depends on the **difference** of (squared) masses, not on the **absolute masses**. Why?

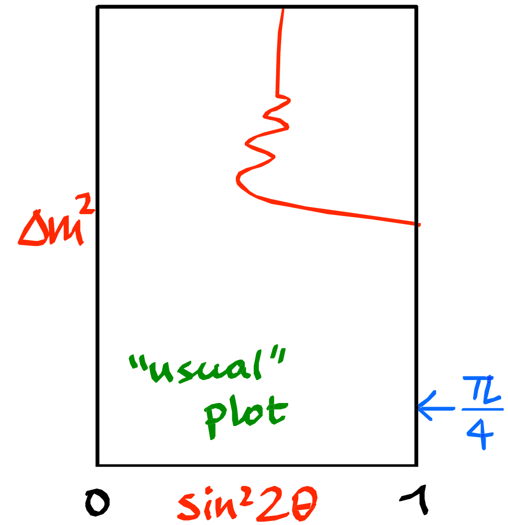
Exercise 4 . Show that: $$\frac{\Delta m^2 L}{4E} = 1.267\left(\frac{\Delta m^2}{\mathrm{eV}^2}\right)\left(\frac{L}{\mathrm{km}}\right)\left(\frac{\mathrm{GeV}}{E}\right)$$

Typical iso-$\langle P_{\alpha\beta} \rangle$ contours



Octant symmetry: $\theta \to \frac{\pi}{2} - \theta$ in $P_{e\mu}$

If 2nd octant folded onto the 1st one :

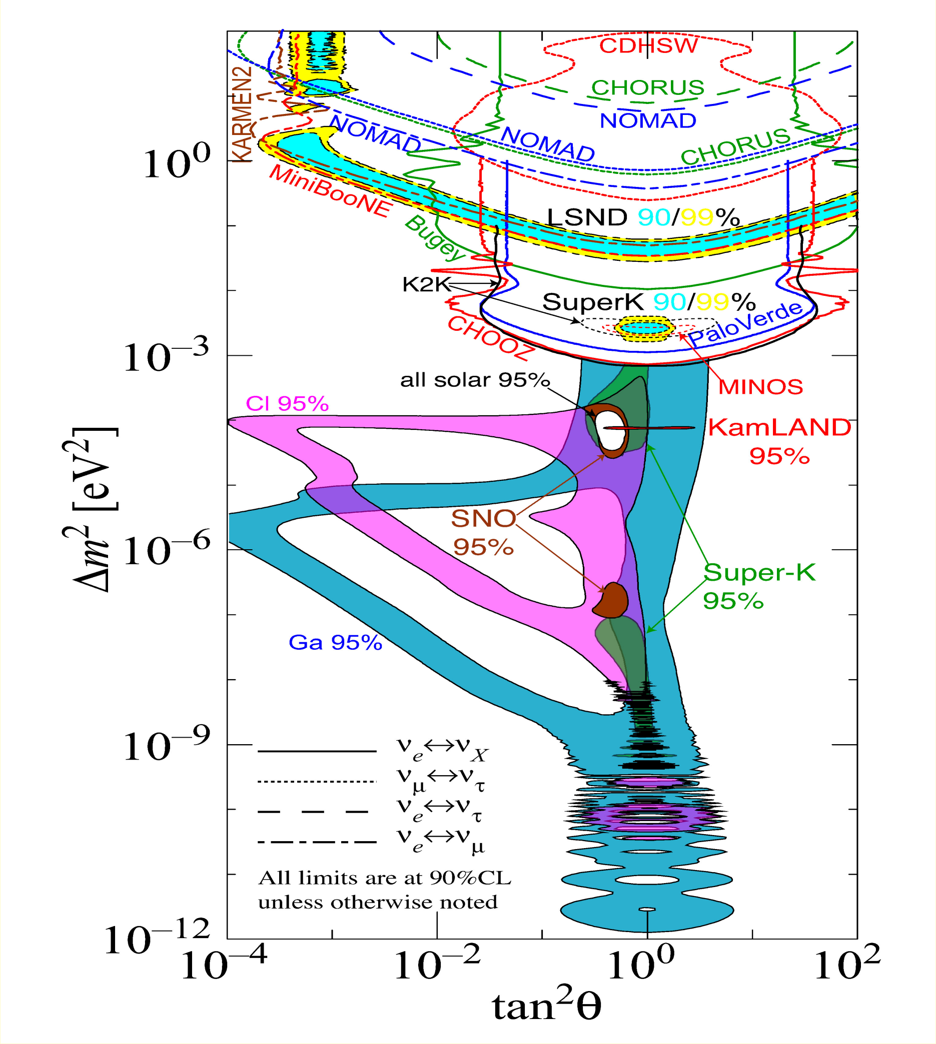


Basically obsolete

In general, better to use (preserve octant-symmetry) $\log \tan^2\theta$ or $\sin^2\theta$

(Note: Octant symmetry broken by 3ν and/or matter effects)

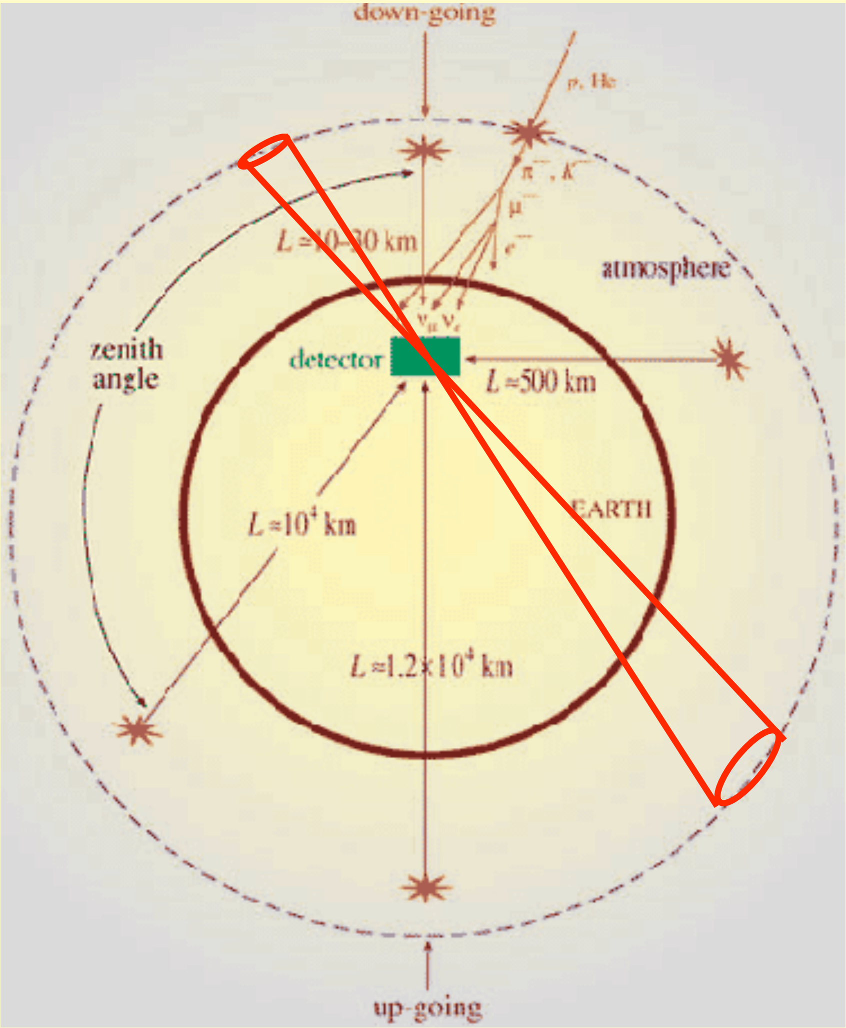
Octant (a)symmetric contours:



[Particle Data Group 2008]

Observation of "effective 2ν" oscillations of atmospheric ν's

Cosmic rays hitting the atmosphere can generate secondary (anti)neutrinos with electron and muon flavor via meson decays.
Energies: E~ 0.1 - 100 GeV. Pathlengths: L~ 10 - 10000 km



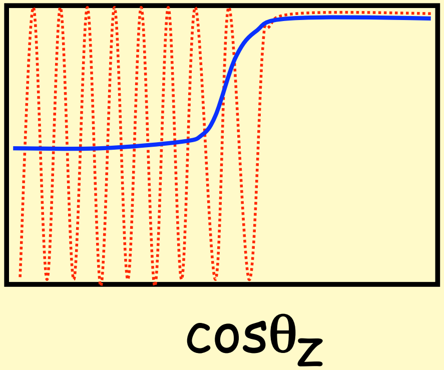
Same ν flux expected from opposite solid angles (up-down symmetry)

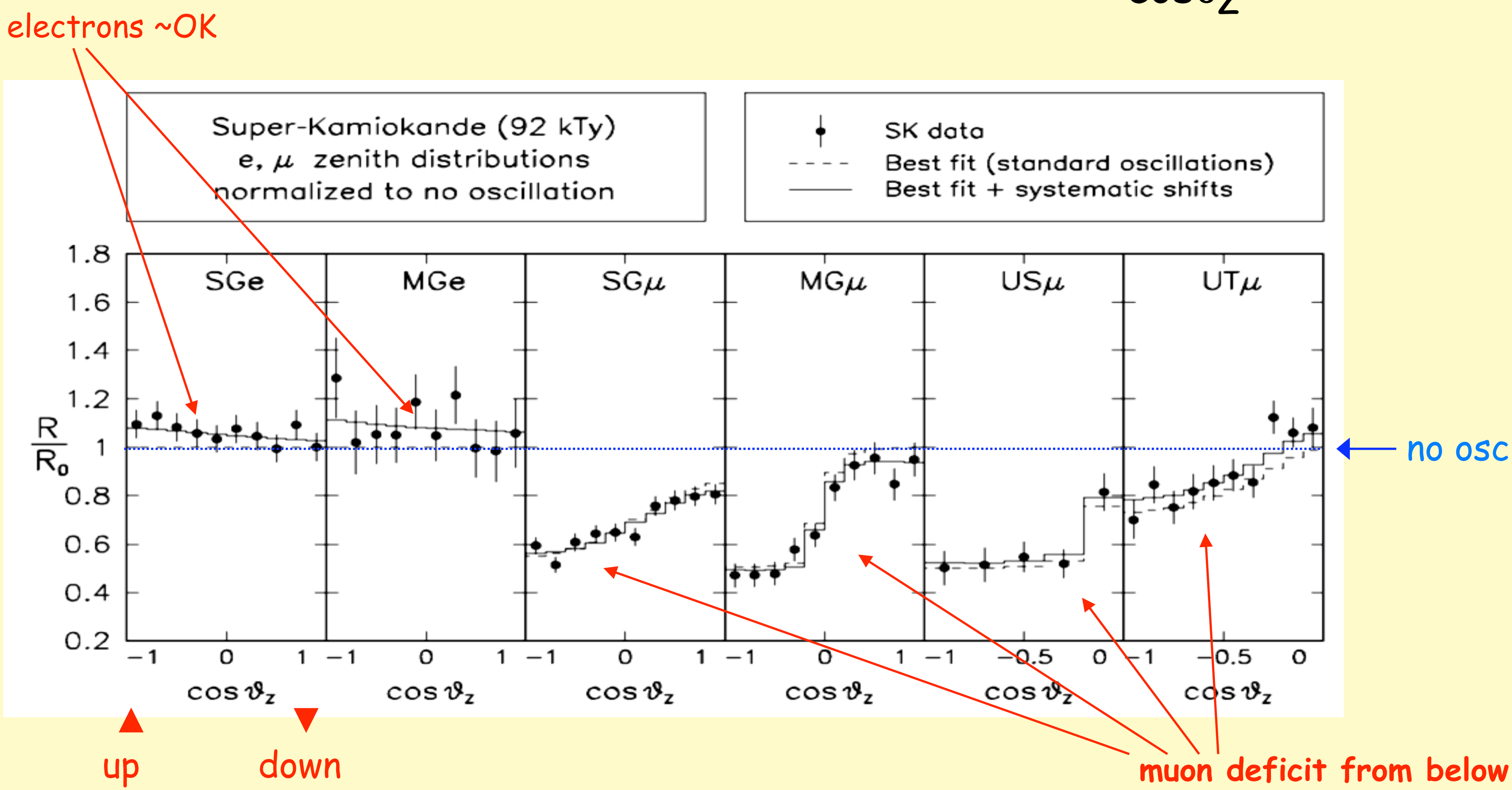
[Flux dilution (~1/r²) is compensated by larger production surface (~r²)]

Should be reflected in symmetry of event zenith spectra, if energy & angle can be reconstructed well enough

Atmospheric neutrinos: Super-Kamiokande

SGe	Sub-GeV electrons
MGe	Multi-GeV electrons
SGμ	Sub-GeV muons
MGμ	Multi-GeV muons
USμ	Upward Stopping muons
UTμ	Upward Through-going muons





Observations over several decades in L/E:

ν_e induced events: ~ as expected

ν_μ induced events: disappearance from below

Interpretation in terms of oscillations:

Channel $\nu_\mu \rightarrow \nu_e$? No (or subdominant)

Channel $\nu_\mu \rightarrow \nu_\tau$? Yes (dominant)

2ν-like approximation works well…

$$P_{\mu\tau} = \sin^2(2\theta)\ \sin^2(\Delta m^2 L/4E_\nu)$$

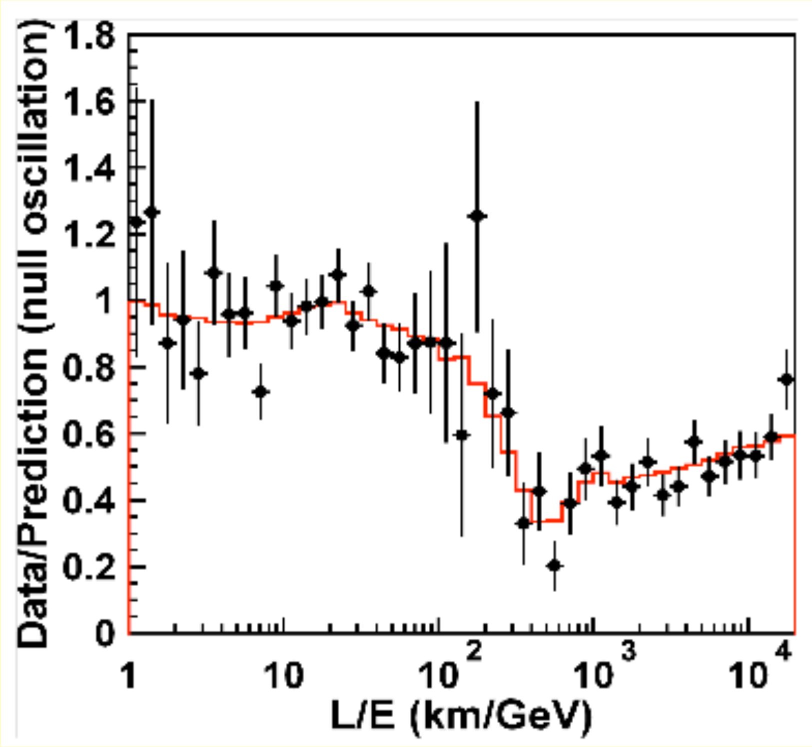
[In this channel, oscillations are ~vacuum-like, despite the presence of Earth matter]

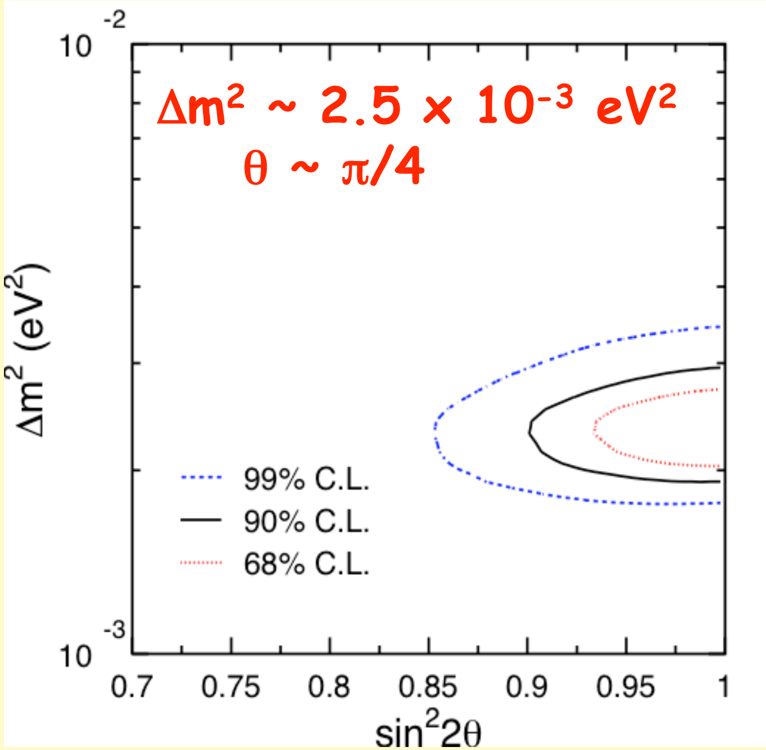
… but where are the “oscillations” ?

Dedicated L/E analysis to "see" half-period of oscillations

1st oscillation dip still visible despite large L & E smearing

Strong constraints on the parameters (Δm^2, θ)

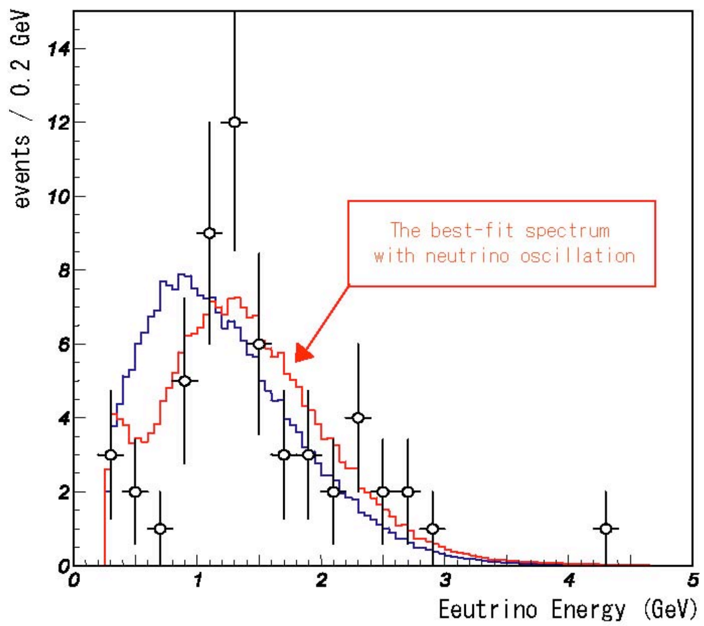




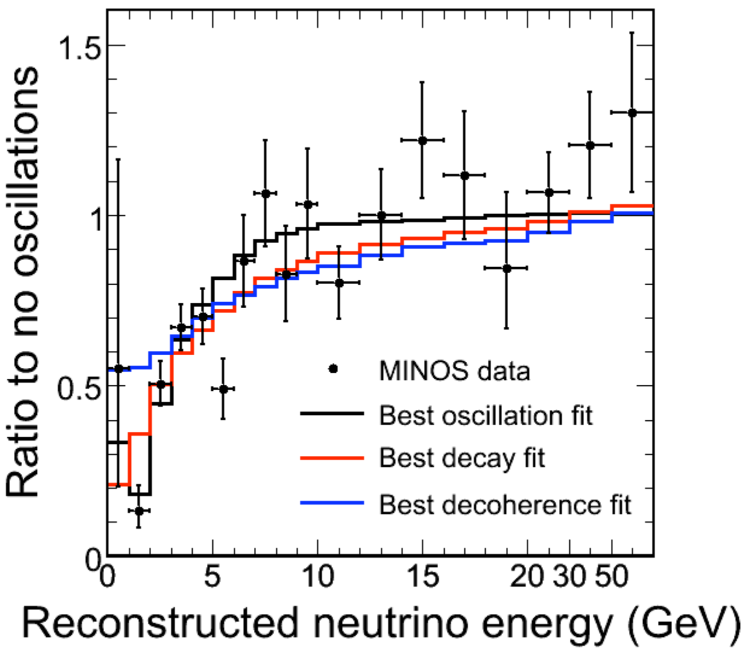
Same mass/mixing parameters confirmed in disappearance mode ($\nu_\mu \rightarrow \nu_\mu$) by other atmospheric expts (MACRO, Soudan2) and by expts with accelerator beams (K2K, MINOS)

Accelerator Results (muon disappearance mode)

K2K



MINOS



1^{st} oscillation dip also observed.

[Exotic explanations without dip (decay, decoherence) disfavored]

Open questions for Δm^2-driven ν_μ oscillations:

The quest for hierarchy and octant: Is the sign of Δm^2 positive ("normal hierarchy") or negative ("inverted hierarchy")? Is θ > or < $\pi/4$?

The quest for ν_τ appearance: We expect dominant $\nu_\mu \rightarrow \nu_\tau$ transitions, but haven't seen the τ flavor directly – the hunt is going on with the CNGS beam. See talks by L. Stanco, A Guglielmi

The quest for ν_e appearance: We haven't seen $\nu_\mu \rightarrow \nu_e$ transitions; are they absent or just suppressed? This is a crucial problem for its implications on leptonic CP violation. See later, & talk by M. Mezzetto

The quest for sterile neutrinos: Besides the known neutrinos $\nu_{e\mu\tau,L}$ (LH, gauge doublets) there might be new "sterile" states $\nu_{s,R}$ (RH, gauge singlets) leading to further disappearance $\nu_{\mu L} \rightarrow (\nu_{s,R})^c$
See talk by C. Giunti

Useful to rephrase some of these questions in 3ν language →

3ν, 1st step: one dominant mass splitting

- 3 flavor and mass states:

$$(\nu_e,\, \nu_\mu,\, \nu_\tau)^T = U\,(\nu_1,\, \nu_2,\, \nu_3)^T$$

Unitary matrix U depends on: 3 rotation angles θ_{ij} + 1 complex CP phase.
Conventionally, same ordering of the CKM quark matrix used for neutrinos:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $c_{ij}=\cos(\theta_{ij})$ etc.

[For antineutrinos: $U \rightarrow U^*$]

- For the 3 masses, let's assume for the moment a single dominant splitting:

$$m_1 \simeq m_2 \quad \text{and} \quad \Delta m^2 = |m_3^2 - m_{1,2}^2|$$

which is a reasonable approx. for all experiments where $\Delta m^2 L/4E \sim O(1)$ namely, atmospheric, long-baseline accelerator, short-baseline reactor expts.

Then, the vacuum oscillation probabilities are generalized as (2ν → 3ν):

$$P_{\alpha\beta} \simeq \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \longrightarrow P_{\alpha\beta} \simeq 4|U_{\alpha 3}|^2|U_{\beta 3}|^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

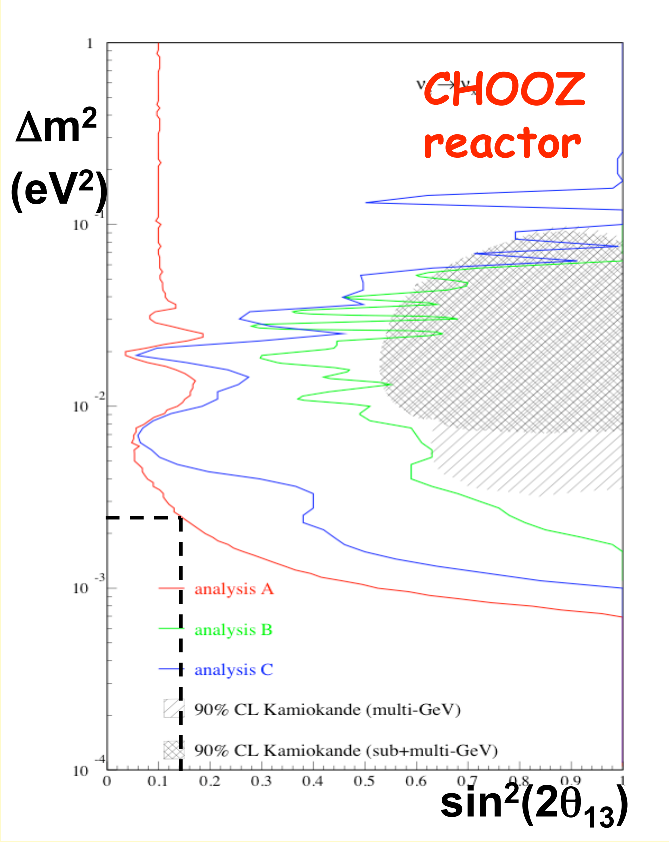
$$P_{\alpha\alpha} \simeq 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \longrightarrow P_{\alpha\alpha} \simeq 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

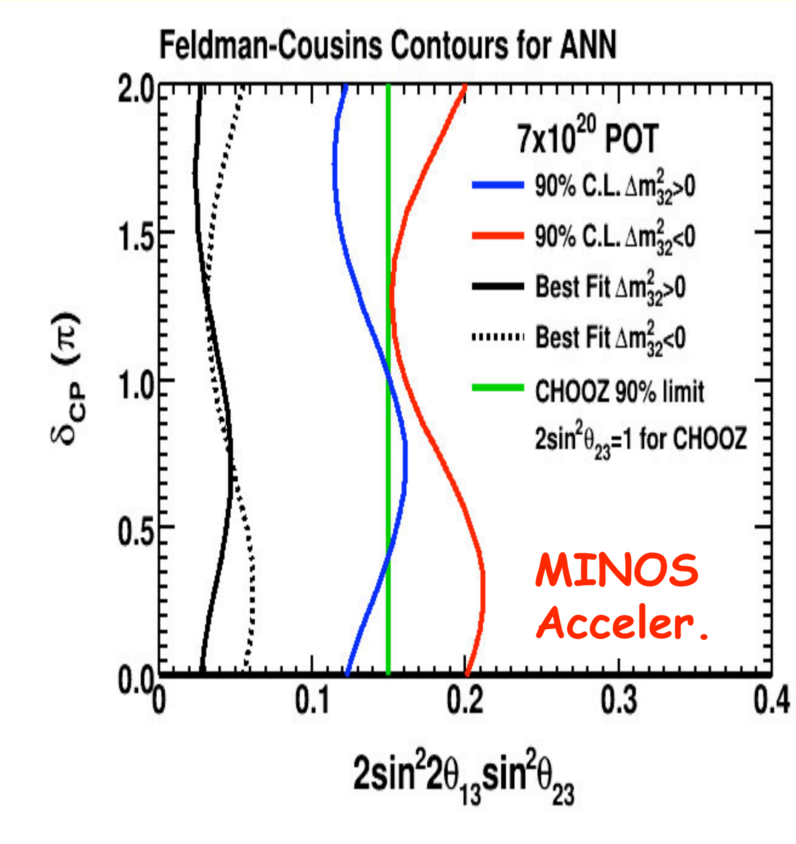
The amplitudes now differ in different oscillation channels, yet they do not depend on the hierarchy or the CP phase.
Also, they do not depend on θ_{12}, due to the assumed degeneracy $m_1 \approx m_2$

In such notation, the previous "$\nu_\mu \to \nu_\tau$" mixing angle is $\theta_{23} \sim \pi/4$, while θ_{13} modulates the oscillation amplitude in the $\nu_e \to \nu_e$ and $\nu_\mu \to \nu_e$ channels where, unfortunately, no signal has been found so far…

$P_{ee} = 1 - \sin^2(2\theta_{13})\sin^2(\Delta m^2 L/4E_\nu)$

$P_{\mu e} = \sin^2\theta_{23}\sin^2(2\theta_{13})\sin^2(\Delta m^2 L/4E_\nu)$

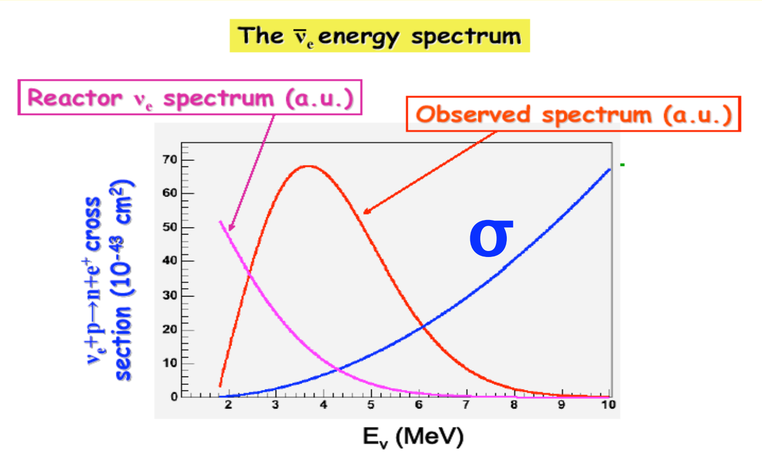




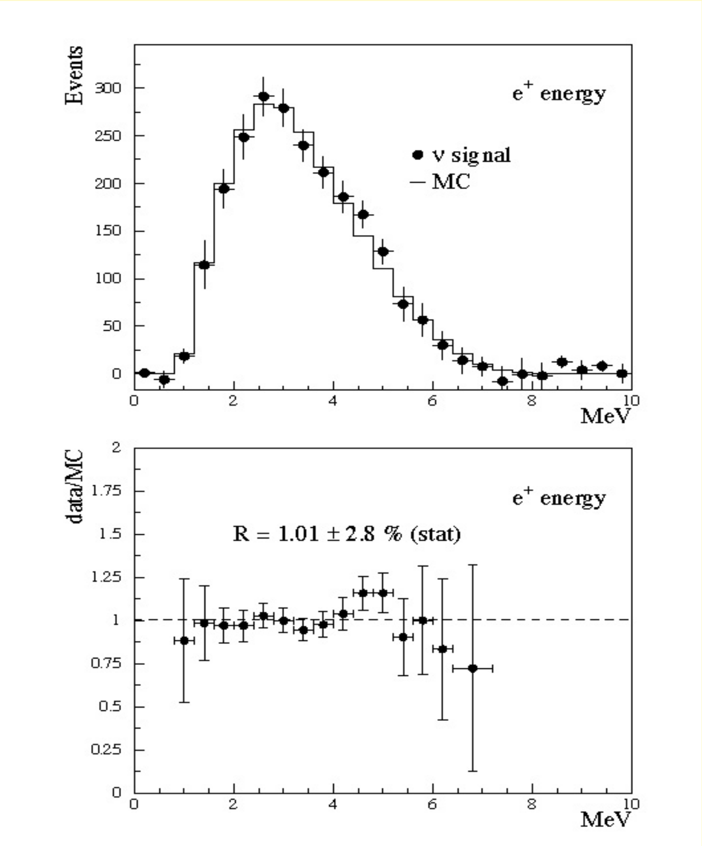
World data consistent with $\sin^2\theta_{13} <$ **few %.**

- More about CHOOZ results (L~1km) -

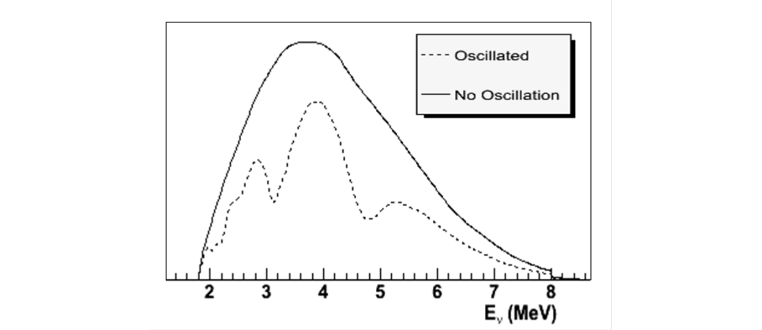
Expected spectrum (no oscill.):



Data: no oscillations within few % error



With oscillations (qualitative):



Future reactor expts: Reduce syst's with near/far detectors.

But: Why hope for θ_{13}>0 after all?

3ν, 2nd step: two mass splittings

We have seen that atmospheric (and long-baseline accelerator) experiments have established the mass splitting of ν_3 with respect to $\nu_{1,2}$, with oscillation parameters:

$$\Delta m^2 = |m_3^2 - m_{1,2}^2| \simeq 2.5 \times 10^{-3}\ \mathrm{eV}^2$$
$$\sin^2\theta_{23} \simeq 0.5$$

We shall see tomorrow that solar and long-baseline reactors, sensitive to much larger L/E, have established the splitting between ν_1 and ν_2 with oscillation parameters:

$$\delta m^2 = m_2^2 - m_1^2 \simeq 8 \times 10^{-5}\ \mathrm{eV}^2$$
$$\sin^2\theta_{12} \simeq 0.3$$

This opens the door to leptonic CP violation, iff $\theta_{13}>0$!

In a full 3ν scenario, a CP violating difference may arise between neutrino and antineutrino oscillation probabilities,

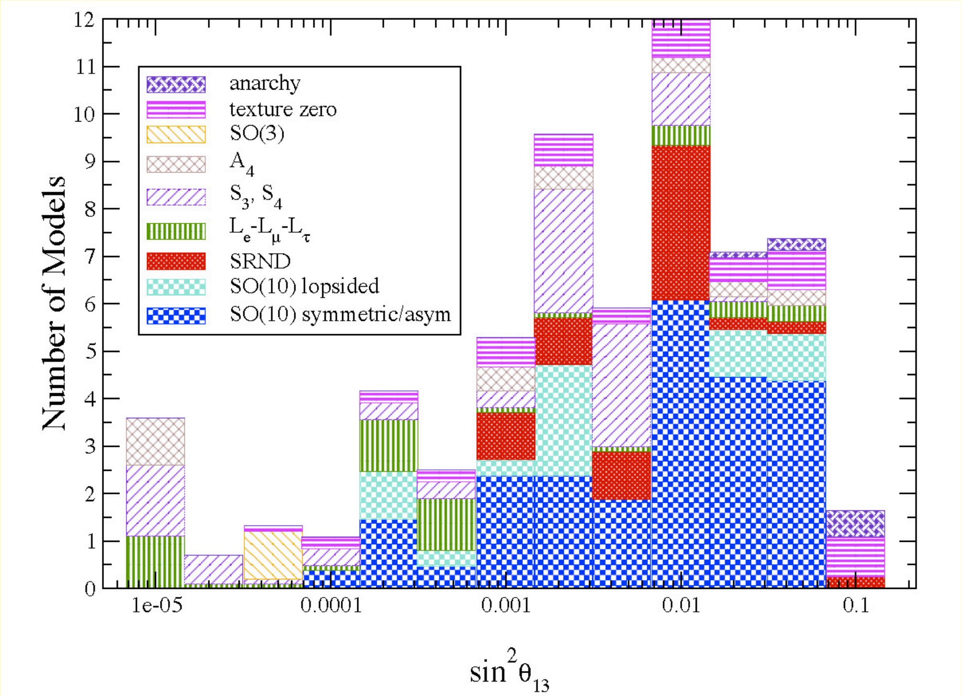
$$P_{\alpha\beta}(\nu) - P_{\alpha\beta}(\bar{\nu}) = 2\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$$
$$\times \sin\left(\frac{\Delta m^2 - \frac{\delta m^2}{2}}{4E}L\right)\sin\left(\frac{\Delta m^2 + \frac{\delta m^2}{2}}{4E}L\right)\sin\left(\frac{\delta m^2}{4E}L\right)$$

provided that:

- $\sin 2\theta_{13}$ is nonzero
- $\sin\delta$ is nonzero
- the oscillation phases are neither too small nor too large

Hunt for θ_{13} crucial in current neutrino research

Also: θ_{13} very important to restrict theoretical models for ν masses

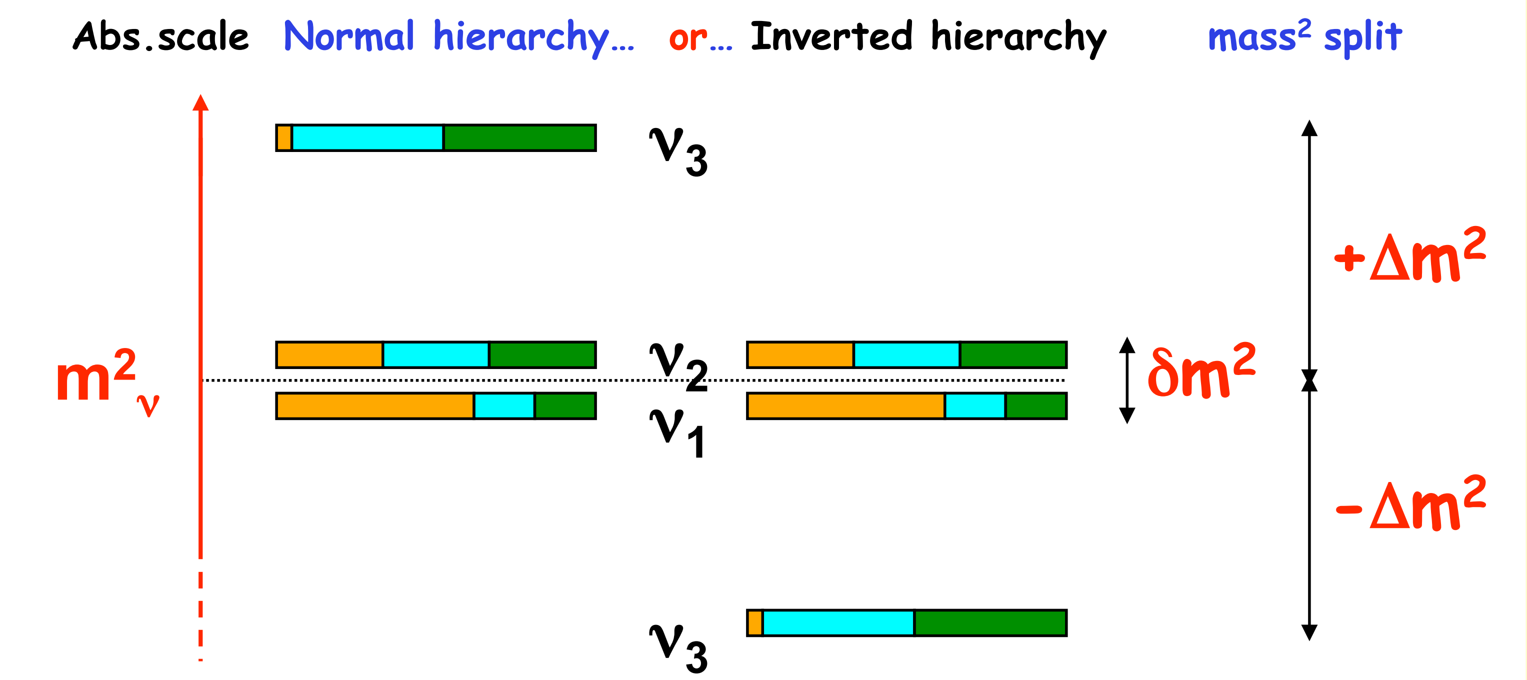


E.g.: CH Albright, 2008, "distribution" of published predictions

See talk by F. Feruglio

3ν mass-mixing overview in just one slide

(here, with 1 digit accuracy). Flavors = e μ τ



$\delta m^2 \sim 8 \times 10^{-5}\ \mathrm{eV}^2$ $\quad$ $\sin^2\theta_{12} \sim 0.3$

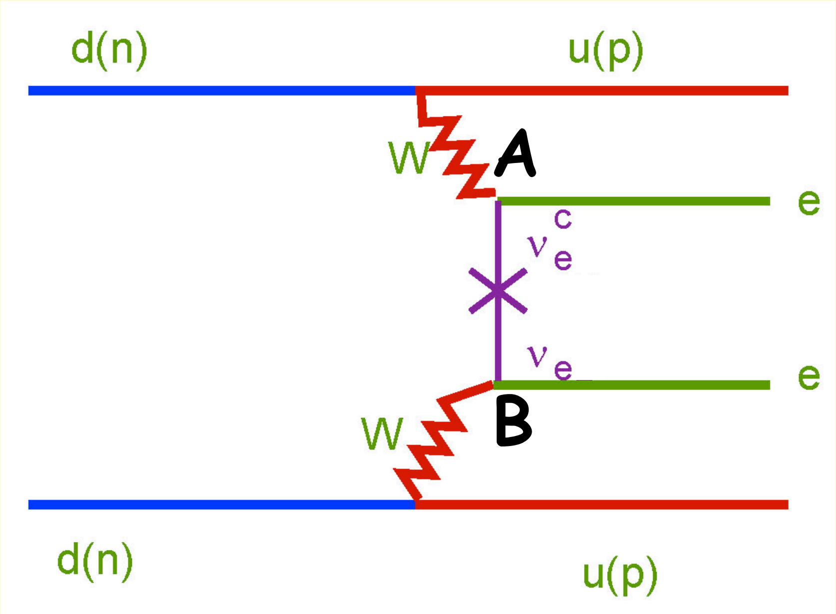
$\Delta m^2 \sim 3 \times 10^{-3}\ \mathrm{eV}^2$ $\quad$ $\sin^2\theta_{23} \sim 0.5$

$m_\nu < O(1)\ \mathrm{eV}$ $\quad$ $\sin^2\theta_{13} < \mathrm{few}\%$

$\mathrm{sign}(\pm\Delta m^2)$ unknown $\quad$ δ (CP) unknown

Implications of 3ν mixing for 0νββ decay

For each mass state ν_i, 0νββ amplitude proportional to:



... mixing of ν_e with ν_i

... mass of ν_i

... mixing of ν_i with ν_e

(times an unknown ν_i phase)

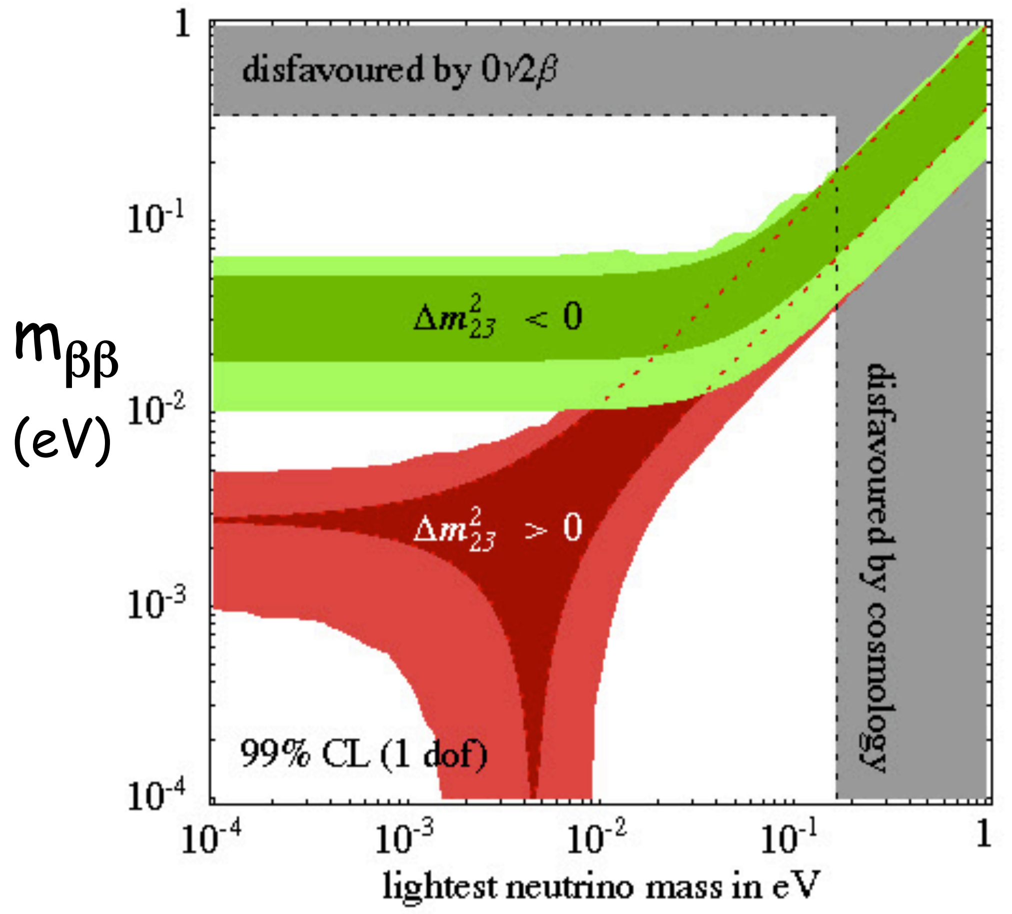
Summing up for three massive neutrinos:

Amplitude ~ "effective Majorana mass"

$$m_{\beta\beta} = \left| c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3} \right|$$

[complex linear combination of masses; c_{ij} = cos θ_{ij} etc.]

Typical plot of $m_{\beta\beta}$ versus lightest neutrino mass:



~degenerate masses

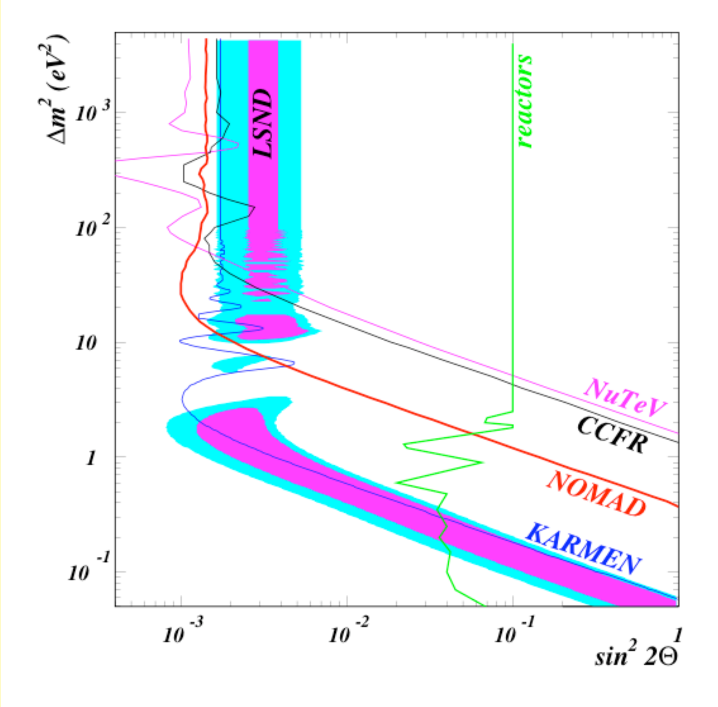
Inverted hierarchy

Normal hierarchy

$$m_{\beta\beta} = \left| c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3} \right|$$

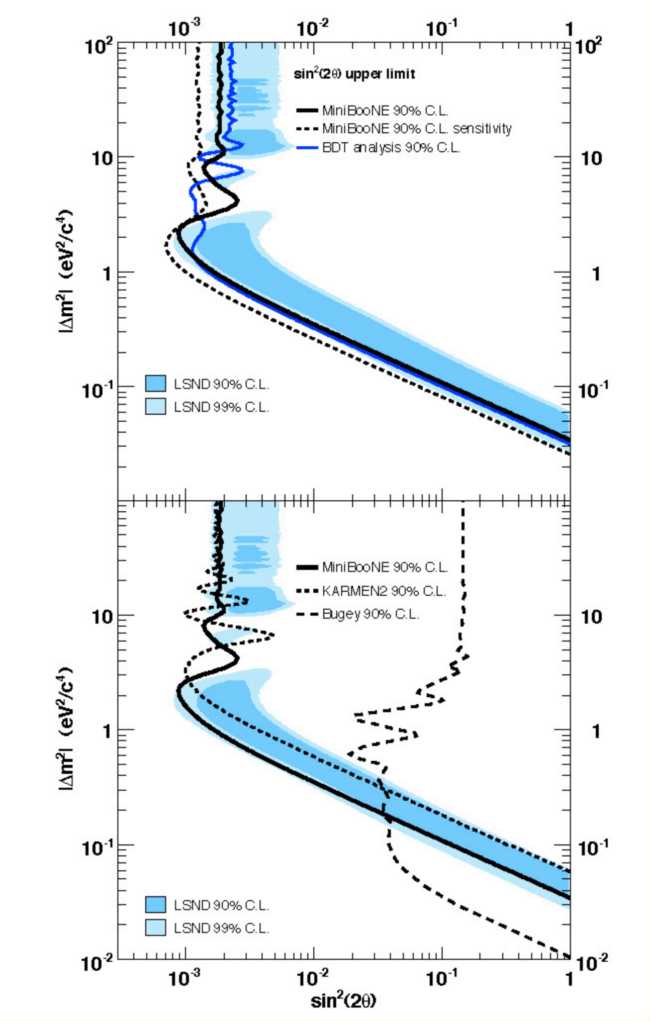
Short baseline accelerator expts: Beyond 3 neutrinos?

The LSND experiment found a signal of possible $\nu_\mu \rightarrow \nu_e$ oscillations at a relatively high ΔM^2 scale of O(0.1-1) eV^2



Large literature on attempts to reconcile LSND with other data, by using new (sterile) neutrino states and/or new neutrino interactions.
No satisfactory model emerged so far. Moreover…

… simplest LSND "oscillations" excluded by a dedicated test experiment, MiniBoone:



But, the MiniBoone data have some new, unexplained anomalies at low energy!

So the LSND/MiniBoone saga may have not yet ended …

Sterile neutrinos and new physics at work? See talk by C. Giunti

RECAP and end of LECTURE I

Poster of the Neutrino Oscillation Workshop 2004 (NOW 2004, Otranto, Italy)

HOMEWORK

Solution 1

- If ν's are Dirac, then $\nu_e \neq \bar{\nu}_e$, and one can attach a leptonic number to the doublets (ν_e, e^-) and $(\bar{\nu}_e, e^+)$, which is conserved in the observed reactions $(\Delta L = 0)$ and would be violated in the other two $(\Delta L = 2)$.

- If ν's are Majorana, then $\nu_e = \bar{\nu}_e$, and we are just naming:

  "ν_e" = LH component of ν state

  "$\bar{\nu}_e$" = RH component of ν state

  The initial "ν_e" is LH, being produced in a weak (β^+) decay. While propagating, it remains dominantly LH, but can develop a small RH component ("$\bar{\nu}_e$") at $\mathcal{O}(m/E)$. Then also the reaction $\bar{\nu}_e + n \to p + e^-$ can take place in principle, but is so suppressed to be practically unobservable. Lepton number violation $(\Delta L = 2)$ is allowed in principle, but suppressed at $\mathcal{O}(m/E)$ in practice.

Solution 2

- Mass basis $\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$ and flavor basis $\begin{pmatrix}\nu_\alpha\\ \nu_\beta\end{pmatrix}$ are related by:

$$\begin{pmatrix}\nu_\alpha\\ \nu_\beta\end{pmatrix} = U\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \qquad \text{with } \Delta m^2 = m_2^2 - m_1^2$$

- Evolution equation in mass basis (mb):

$$i\frac{d}{dt}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \mathcal{H}_{mb}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \qquad \leftarrow \text{Schrödinger eq. in natural units } (\hbar = c = 1)$$

where the Hamiltonian is simply

$$\mathcal{H}_{mb} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix} \simeq \begin{pmatrix}p+\frac{m_1^2}{2E} & 0\\ 0 & p+\frac{m_2^2}{2E}\end{pmatrix} = \underbrace{\left(p+\frac{m_1^2+m_2^2}{4E}\right)\mathbb{1}}_{\propto \mathbb{1}} + \underbrace{\begin{pmatrix}-\frac{\Delta m^2}{4E} & 0\\ 0 & +\frac{\Delta m^2}{4E}\end{pmatrix}}_{\text{Traceless}}$$

Final results do not depend on the part proportional to $\mathbb{1}$ - check it.
(Reason: it gives an overall phase which disappears in observable real quantities). So we take:

$$\mathcal{H}_{mb} = \frac{\Delta m^2}{4E}\begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}$$

Solution 2 (ctd)

- Evolution operator in mass basis:

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_t = S_{mb} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_0 \quad \text{where}$$

$$S_{mb} = e^{-i\mathcal{H}_{mb} t} \simeq e^{-i\mathcal{H}_{mb} x} = \begin{pmatrix} e^{i\frac{\Delta m^2}{4E}x} & 0 \\ 0 & e^{-i\frac{\Delta m^2}{4E}x} \end{pmatrix}$$

$\leftarrow$ $x \simeq t$ for ultrarelativistic neutrinos

- Evolution operator in flavor basis (fb):

$$S_{fb} = U S_{mb} U^T$$

$$= \cos\left(\frac{\Delta m^2 x}{4E}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\Delta m^2 x}{4E}\right)\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

- Amplitudes for flavor transitions:

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}_t = S_{fb} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}_0 \rightarrow$$

Off-diagonal elements of S_{fb} give amplitudes for $\nu_\alpha \to \nu_\beta$ and $\nu_\beta \to \nu_\alpha$

Solution 2 (ctd)

- Probability of flavor transition is the square modulus of the amplitude:

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) = \left| -i \sin 2\theta \sin\left(\frac{\Delta m^2 x}{4E}\right)\right|^2$$

$$= \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 x}{4E}\right) \qquad (x = L \text{ in the lecture}).$$

- The diagonal elements of the evolution operator would give the "flavor survival" amplitude. Check that

$$1 - P(\nu_\alpha \to \nu_\beta) = P(\nu_\alpha \to \nu_\alpha) = P(\nu_\beta \to \nu_\beta).$$

$$\to \quad P(\nu_\alpha \to \nu_\alpha) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

Solution 3

Oscillations depend only on the difference of phases, and thus of neutrino energies. Indeed, the results do not change by an overall shift of the Hamiltonian:

$$H \to H + \text{const} \cdot \mathbb{1}$$

Since the zero-point energy is irrelevant in this context, the absolute neutrino mass scale is unobservable (in oscillation searches).

Solution 4

$\hbar c = 197.327 \text{ MeV} \cdot \text{fm} = 1$ in natural units.

Therefore: $1 \text{ MeV} \cdot 1 \text{ m} = 5.0677 \times 10^{12}$

Then:

$$\frac{\Delta m^2 L}{4E} = \frac{1}{4}\left(\frac{\Delta m^2}{eV^2}\, eV^2\right)\left(\frac{L}{m} \cdot m\right)\left(\frac{MeV}{E} \cdot \frac{1}{MeV}\right)$$

$$= \frac{1}{4}\left(\frac{1 eV^2 \cdot 1 m}{1 \text{ MeV}}\right)\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{m}\right)\left(\frac{MeV}{E}\right)$$

$$\frac{1}{4}\frac{eV^2 m}{MeV} = \frac{1}{4} \times 10^{-12}\, \frac{MeV^{\cancel{2}} \cdot 1m}{1\, \cancel{MeV}} = \frac{10^{-12}}{4}(MeV \cdot m) = 0.25 \times 10^{-12} \times 5.0677 \times 10^{12} = 1.267$$

$$\frac{\Delta m^2 L}{4E} = 1.267\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{m}\right)\left(\frac{MeV}{E}\right) = 1.267\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{km}\right)\left(\frac{GeV}{E}\right)$$

Appendix on Majorana mass terms

Dirac and Majorana mass terms (1 family)

- Dirac mass terms are of the form $m\bar{\psi}\psi$ (4 dof ψ)
- Majorana " " " " " $\frac{1}{2}m\bar{\psi}\psi$ (2 dof ψ)

Three possibilities:

Dirac : $\psi = \psi_L + \psi_R \rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$

Majorana (L) : $\psi = \psi_L + \psi_L^c \rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L$

Majorana (R) : $\psi = \psi_R + \psi_R^c \rightarrow \bar{\psi}\psi = \bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R$

Most general mass term for <u>one</u> neutrino family:

$$m_D\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) + \frac{1}{2}m_L\left(\bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L\right) + \frac{1}{2}m_R\left(\bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R\right)$$

[Last two terms absent for charged fermions.]

- Previous mass term can be rewritten as:

$$\frac{1}{2}\left[\bar{\psi}_L + \bar{\psi}_L^c,\ \bar{\psi}_R + \bar{\psi}_R^c\right]\begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}\begin{bmatrix} \psi_L + \psi_L^c \\ \psi_R + \psi_R^c \end{bmatrix}$$

- Diagonalization provides fields with definite masses. (If mass < 0, redefine field $\psi \to \gamma_5 \psi$ so that $m \to -m$)

- Since the basis fields $(\overset{(-)}{\psi}_L + \overset{(-)}{\psi}{}_L^c)$ and $(\overset{(-)}{\psi}_R + \overset{(-)}{\psi}{}_R^c)$ are Majorana, diagonalization will generally produce mass eigenvectors which are also Majorana

Diagonalize $M = \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix}$

$$M = \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \qquad T = \mathrm{Tr}\, M = m_L + m_R$$
$$D = \det M = m_L m_R - m_D^2$$

Eigenvalues : $m_\pm = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$

Diagonalization angle : $\sin 2\theta = \dfrac{m_D}{\sqrt{T^2-4D}} \qquad \cos 2\theta = \dfrac{m_L - m_R}{\sqrt{T^2-4D}}$

(not a mixing angle!)

$$\begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} = \begin{bmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix}$$

Eigenvectors

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1' & v_2' \end{bmatrix} \begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The see-saw mechanism

Many extensions of the Standard Model predict the existence of singlet neutrinos (ν_R).

E.g., in the $\underline{16}$ representation of SO(10):

$$\begin{pmatrix} u_L & u_L & u_L & \nu_L \\ d_L & d_L & d_L & e_L \\ u_R & u_R & u_R & \boldsymbol{\nu_R} \\ d_R & d_R & d_R & e_R \end{pmatrix}$$

→ can get a Majorana mass term $\sim m_R(\bar{\nu}_R^c \nu_R + \bar{\nu}_R \nu_R^c)$, where m_R is presumably a large mass scale characterizing the SM extension.

For $m \ll M$, diagonalization of $\begin{bmatrix} 0 & m \\ m & M \end{bmatrix}$ gives:

Eigenvectors (ν fields)	Eigenvalues (masses)
$\nu_{heavy} \cong (\nu_R + \nu_R^c) + \frac{m}{M}(\nu_L + \nu_L^c)$	M
$\nu_{light} \cong -(\nu_L + \nu_L^c) + \frac{m}{M}(\nu_R + \nu_R^c)$	$(-)\frac{m^2}{M} \ll m$

← see-saw

The light state is active (contains ν_L) and has a very small mass $\sim m^2/M$

Presumably: $m \sim \mathcal{O}(m_{quarks}, m_{leptons})$

$M \sim \mathcal{O}(\Lambda_{beyond\ SM})$

The see-saw mechanism might explain the smallness of neutrino masses