Introduction to Neutrino Mixing and Neutrino Oscillations

Carlo Giunti
INFN, Sezione di Torino
giunti@to.infn.it

Brief History of Standard Neutrino Properties

1930: Pauli proposes the existence of a new particle emitted in β decay to solve the problems of spin and statistics of nuclei and continuous β spectrum:

> ... I have hit upon a desperate remedy to save the "exchange theorem"[a] of statistics and the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin 1/2 and obey the exclusion principle and which further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. ...

[a] This reads: exclusion principle (Fermi statistics) and half-integer spin for an odd number of particles; Bose statistics and integer spin for an even number of particles. (W. Pauli) [Building blocks of nuclei: protons and electrons $\Rightarrow$ symmetry character of nuclei determined by atomic number. Experiment $\Rightarrow$ symmetry character of nuclei determined by atomic mass.]

1932: Chadwick discovers the neutron $\Rightarrow$ problem of spin and statistics of nuclei solved.

1933-34: Fermi introduces the name neutrino in his four-fermion theory of β decay, formulated in analogy with QED:

$$\mathcal{L}_{\mathrm{QED}} = -e\, j_\mu^{\mathrm{em}}\, A^\mu \,, \qquad \underbrace{j_\mu^{\mathrm{em}} = \overline{p}\,\gamma_\mu\, p - \overline{e}\,\gamma_\mu\, e}_{\text{electromagnetic (neutral) current}}$$

$$\Downarrow$$

$$\mathcal{L}_\beta = -\frac{G_F}{\sqrt{2}}\, j_\mu\, j^{\mu\dagger} \,, \qquad \underbrace{j_\mu = \overline{p}\,\gamma_\mu\, n + \overline{\nu}\,\gamma_\mu\, e}_{\text{weak charged current}}$$

1937: Anderson & Neddermeyer discover the muon, that is a heavy version of the electron ($m_\mu = 106\,\mathrm{MeV}$).

1947: Neutrino is much lighter than electron:

$$m_\nu \lesssim 500\,\mathrm{eV}$$

1947: Pontecorvo proposes the universality of weak interactions (Fermi theory) of electron and muons.

1948: Puppi and others extend the universality of weak interactions to all particles.

1956: Cowan & Reines detect (electron) antineutrinos produced in a nuclear reactor through the "inverse β decay reaction"

$$\bar{\nu} + p \to n + e^{+}$$

Confirmation of Pauli hypothesis and Fermi theory.

1956: Lee & Yang propose parity[a] violation in weak interactions (as a solution of the "ϑ-τ puzzle").

1957: Wu *et al.* measure maximal parity violation in β decays of polarized ^{60}Co nuclei.

1958: Goldhaber *et al.* find that the (electron) neutrino has negative helicity: $h(\nu) = -1$, $h(\bar{\nu}) = +1$ [in β decays $h(e^{-}) = -v/c$, $h(e^{+}) = +v/c$] $\Rightarrow$ $V - A$ coupling[b]:

$$j_{\mu} = \overline{\nu}\,(\gamma_{\mu} - \gamma_{\mu}\gamma_{5})\,e$$

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[a] Invariance under spatial reflections.

[b] We use the definition $\gamma_5 \equiv \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

1957: Landau, Lee & Yang and Salam formulate the two-component theory of massless neutrinos.

$$V - A \text{ coupling:} \quad j_\mu = \overline{\nu}\,\gamma_\mu\,(1-\gamma_5)\,e = 2\,\overline{\nu_L}\,\gamma_\mu\,e_L$$

$$\nu_L \equiv \frac{1-\gamma_5}{2}\,\nu \quad \text{left-handed chirality} \quad \gamma_5\,\nu_L = -\nu_L$$

$$\text{Chiral representation:} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nu = \underbrace{\begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}}_{\text{four components}} \quad \Rightarrow \quad \nu_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ \chi_L \end{pmatrix}}_{\text{two components}}$$

Weak interactions involve only two of the four components of the Dirac neutrino field!

$$\text{Four components:} \quad \left\{ \begin{array}{c} (\text{particle}+\text{antiparticle}) \\ \otimes \\ (\text{positive}+\text{negative helicity}) \end{array} \right.$$

Dirac Equation:

$$(i\gamma^\mu \partial_\mu - m)\,\nu = 0 \quad \Rightarrow \quad \big(i\gamma^0 \partial_0 + i\,\underbrace{\gamma^k \partial_k}_{\vec{\gamma}\cdot\vec{\nabla}} - m\big)\,\nu = 0$$

Chiral representation[a]: $\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

$$\begin{pmatrix} -m & i\,(-\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i\,(-\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} = 0$$

Two equations coupled by mass:

$$\begin{cases} i\,(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\chi_L = m\,\chi_R \\ i\,(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\,\chi_R = m\,\chi_L \end{cases}$$

$m = 0 \Rightarrow \chi_R$ is not needed! $\Rightarrow$ two components!

$$\boxed{\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)\chi_L = 0}$$ Weyl Equation (1929)

Rejected by Pauli because parity violating!

(Physically $\partial_0 + \vec{\sigma}\cdot\vec{\nabla}$ and $\partial_0 - \vec{\sigma}\cdot\vec{\nabla}$ are equivalent because the sign of the $\vec{\sigma}$ matrices is arbitrary. Once the sign is fixed, they lead to opposite helicities.)

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[a] γ matrices are defined by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Chirality and Helicity

$$\left(\partial_0 - \vec{\Sigma}\cdot\vec{\nabla}\right)\nu_L(x) = 0 \quad \text{massless chiral field} \qquad \left(\vec{\Sigma} = \left(\begin{smallmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{smallmatrix}\right)\right)$$

Fourier expansion:

$$\nu_L(x) \propto \int \mathrm{d}^3p \sum_{h=\pm1}\left[b_p^{(h)} u_L^{(h)}(p) e^{-ip\cdot x} + d_p^{(h)\dagger} v_L^{(h)}(p) e^{ip\cdot x}\right]$$

Wave function:

$$\nu_L^{(h)}(x,p) = \langle 0|\nu_L(x)|p,h\rangle \propto u_L^{(h)}(p) e^{-ip\cdot x} \leftarrow -iEt+i\vec{p}\cdot\vec{x}$$

$$\left(\partial_0 - \vec{\Sigma}\cdot\vec{\nabla}\right)\nu_L^{(h)}(x,p) = 0 \Rightarrow \left(-iE - i\vec{\Sigma}\cdot\vec{p}\right)\nu_L^{(h)}(x,p) = 0$$

$$E = |\vec{p}| \Rightarrow \underbrace{\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}}_{\text{Helicity}}\nu_L^{(h)}(x,p) = -\nu_L^{(h)}(x,p) \Rightarrow h = -1$$

Massless two-component neutrinos described by ν_L have negative helicity and antineutrinos have positive helicity!

$$\nu_L(x) \propto \int \mathrm{d}^3p \left[b_p^{(-)} u_L^{(-)}(p) e^{-ip\cdot x} + d_p^{(+)\dagger} v_L^{(+)}(p) e^{ip\cdot x}\right]$$

Massless fermion $\Rightarrow$ Chirality $=$ Helicity

But for massive fermions chirality $\neq$ helicity!

Helicity in different frames

Helicity is conserved: $[\widehat{h}\,,\,\widehat{H}] = 0$

Good quantum number for classification of states!

But in general helicity is not Lorentz invariant:

$$\left.\begin{array}{c} \vec{p}\,,\ \vec{s} \\ h=-1 \end{array}\right\} \xrightarrow{\text{boost}} \left\{\begin{array}{c} -\vec{p}\,,\ \vec{s} \\ h=+1 \end{array}\right.$$

Massive fermion: both helicity states must exist:

$$\mathfrak{f}(h=-1) \xrightarrow{\text{boost}} \mathfrak{f}(h=+1)$$

Massless fermion: boost is impossible:

Helicity is Lorentz invariant!

Neutrino can be left-handed only if massless!

1958: Feynman & Gell-Mann and Sudarshan & Marshak extend the $V-A$ form of weak interactions to all particles ($V-A$ theory).

1962: Lederman & Steinberger & Schwartz realize an experiment proposed by Pontecorvo in 1960 and prove that $\nu_\mu \neq \nu_e$. Weak interactions conserve the lepton numbers L_e, L_μ:

$$j_\alpha^{\mathrm{CC}} = 2\left(\overline{\nu_{eL}}\,\gamma_\alpha\, e_L + \overline{\nu_{\mu L}}\,\gamma_\alpha\,\mu_L\right)$$

1967: Weinberg and Salam formulate the Standard Model of Electro-Weak Interactions applying the Higgs mechanism to the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge model proposed by Glashow in 1961. Charged current weak interactions are mediated by the heavy $W^\pm$ vector bosons ($m_W \simeq 80\,\mathrm{GeV}$):

$$\mathcal{L}_{\mathrm{CC}} = -m_W \sqrt{\frac{G_F}{\sqrt{2}}}\, j_\mu^{\mathrm{CC}}\, W^\mu + \mathrm{h.c.}$$

The standard model predicts the existence of weak neutral current interactions mediated by the heavy Z vector boson ($m_Z \simeq 90\,\mathrm{GeV}$):

$$\mathcal{L}_{\mathrm{NC}} = -\sqrt{2}\, m_Z \sqrt{\frac{G_F}{\sqrt{2}}}\, j_\mu^{\mathrm{NC}}\, Z^\mu\,, \qquad \underbrace{j_\mu^{\mathrm{NC}} = \sum_\ell \overline{\nu_{\ell L}}\,\gamma_\mu\,\nu_{\ell L}}_{\text{neutral current}}$$

1968: Solar ^{8}B neutrinos ($^{8}\mathrm{B} \to {}^{8}\mathrm{Be} + e^{+} + \nu_e$) are detected in the radiochemical Homestake experiment (Davis) through the reaction

$$\nu_e + {}^{37}\mathrm{Cl} \to {}^{37}\mathrm{Ar} + e^{-}$$

proposed by Pontecorvo in 1946. Important confirmation of stellar energy generation by thermonuclear reactions. But the measured flux is about $1/3$ of the one predicted by the Standard Solar Model (Bahcall) $\Rightarrow$ Solar Neutrino Problem!

1973: Weak neutral currents are discovered at CERN (Gargamelle: $\nu_\mu + N \to \nu_\mu + X$ and $\bar{\nu}_\mu + N \to \bar{\nu}_\mu + X$).

1975: Perl *et al.* discover the tau, that is a heavy version of the electron and muon ($m_\tau = 1.8\,\mathrm{GeV}$).

Standard Model

$$\Downarrow$$

$$\boxed{\mathbb{L}_e = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \,, \quad \mathbb{L}_\mu = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \,, \quad \mathbb{L}_\tau = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}}$$

$$\Downarrow$$

ν_τ + conserved tau lepton number L_τ

1983: The experiments UA1 and UA2 at CERN reveal the $W^{\pm}$ and Z bosons.

1987: The Kamiokande and IMB experiments (constructed in $\sim$1980 as proton decay experiments and improved in $\sim$1985 to be also neutrino telescopes) detect about a dozen of neutrinos produced by Type II supernova SN1987A in the Large Magellanic Cloud, at a distance of about 50 kpc. Confirmation of main features of supernova theory.

1989: LEP experiments measuring the invisible width of the Z boson ($Z \to \sum_{\ell} \nu_{\ell} \bar{\nu}_{\ell}$) determine with high accuracy that the number of light neutrino flavors is 3: ν_e, ν_μ, ν_τ. The present number is 3.00 ± 0.06 (a Standard Model fit of all LEP data yields 2.994 ± 0.012).

2000: The DONUT experiments detects τ's produced by charged-current interactions of ν_τ's.

Common belief: DONUT has first direct evidence of the existence of ν_τ.

Wrong!: From basic principles of QFT the particle produced with τ must be able to produce τ!

DONUT neutrino beam contains also ν_e, ν_μ that generate electrons and muons in the detector.

DONUT cannot prove that the particle produced with τ does not generate electrons and muons in the detector.

From DONUT alone, the particle produced with τ could be ν_e or ν_μ.

Then, has ν_τ been experimentally discovered?.

Yes!: From measured spectrum of τ decay into leptons and negative results of $\nu_\mu \to \nu_\tau$ and $\nu_e \to \nu_\tau$ experiments (FNAL E531, CHARM, CCFR, NOMAD, CHORUS), which have shown that the particle which produces τ (that from QFT is also the particle produced with τ) is different from ν_e and ν_μ.

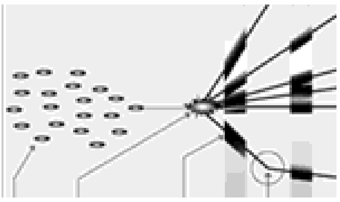
In practice DONUT has proved that it is technically possible to detect ν_τ.

News: July 2000

Back to Archive Article 9 of 11

Tau neutrino identified at last

[25 Jul 2000] The first direct evidence for the existence of the tau neutrino has finally been found. The tau neutrino is the last of the particles in the Standard Model of elementary particles to be detected. An international collaboration of 54 physicists at Fermilab in Illinois, where the news was announced on 21 July, made the discovery after a three-year analysis of data from the Direct Observation of the Nu Tau (DONUT) experiment. Neutrinos are extremely difficult to observe because they have no electric charge and barely interact with matter.



Detecting a tau neutrino

The DONUT team fired an intense beam of neutrinos, which they expected to contain tau neutrinos, at a target consisting of iron plates with layers of emulsion sandwiched between them. One in a million million tau neutrinos interacted with an iron nucleus to produce a tau lepton, which subsequently decayed leaving a characteristic track in the emulsion. The team isolated just four tracks containing the telltale kink of the tau neutrino track, out of a possible six million.

Far from completing the picture for neutrino physics, the new discovery poses many more questions, such as whether the tau neutrino has mass, and whether it interacts differently to the other neutrinos. The answers to these questions will have major implications for our understanding of the evolution of the universe, and international efforts to find them are already underway.

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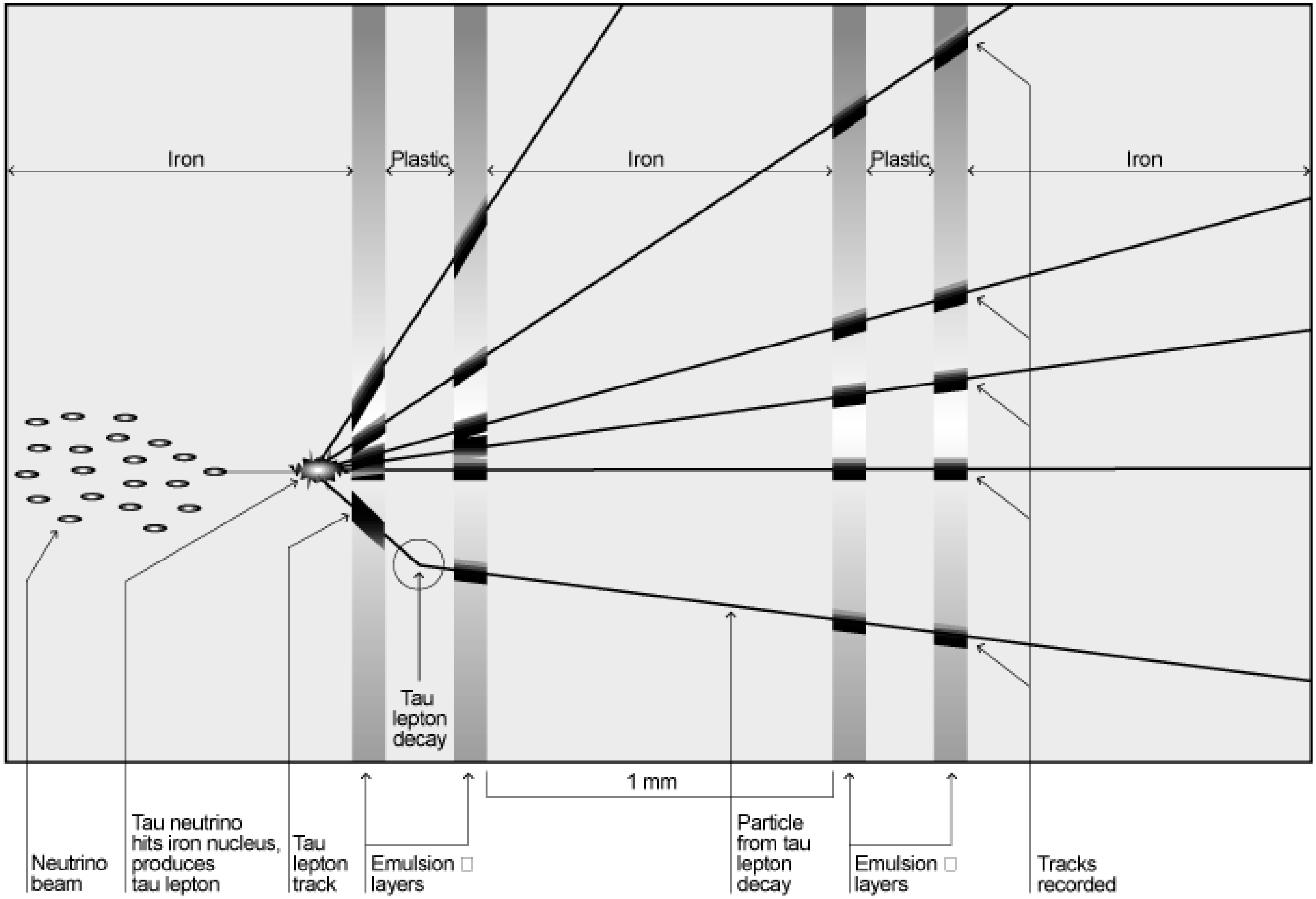
Back to Archive Article 9 of 11



Of one million million tau neutrinos crossing the DONUT detector, scientists expect about one to interact with an iron nucleus.

Long History of Exotic Neutrino Properties

$$\left(\begin{array}{c} \text{“Exotic”} \quad \equiv \quad \text{“beyod the Standard Model} \\ \text{of massless neutrinos”} \\ \text{But what is exotic today may be standard tomorrow!} \end{array} \right)$$

1937: Majorana discovers the possibility of existence of truly neutral fermions.

Charged Fermion (electron) + Electromagnetic Field[a]:

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\,\psi = 0 \qquad \text{particle}$$

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)\,\psi^c = 0 \qquad \text{antiparticle}$$

It is clearly forbidden that $\psi^c = \psi$.

Neutral Fermion (neutrino) + Electromagnetic Field:

$$(i\gamma^\mu \partial_\mu - m)\,\nu = 0 \qquad \text{particle}$$

$$(i\gamma^\mu \partial_\mu - m)\,\nu^c = 0 \qquad \text{antiparticle}$$

ν and ν^c obey to the same equation $\Rightarrow$ they can be equal:

$\boxed{\nu^c = \nu}$ Majorana condition particle=antiparticle

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[a] $\psi^c = \mathcal{C}\overline{\psi}^T$, where $\mathcal{C}$ is the charge-conjugation matrix, such that $\mathcal{C}\,\gamma_\mu^T\,\mathcal{C}^{-1} = -\gamma_\mu$, $\mathcal{C}^\dagger = \mathcal{C}^{-1}$, $\mathcal{C}^T = -\mathcal{C}$. Moreover, $\mathcal{C}\,\gamma_5^T\,\mathcal{C}^{-1} = \gamma_5$.

In the chiral representation

$$\nu = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} , \quad \nu^c = \begin{pmatrix} -i\sigma^2 \chi_L^* \\ i\sigma^2 \chi_R^* \end{pmatrix}$$

Four independent components

Majorana Fermion: $\nu^c = \nu$

$$\left. \begin{array}{l} \chi_R = -i\sigma^2 \chi_L^* \\ \chi_L = i\sigma^2 \chi_R^* \end{array} \right\} \text{equivalent} \Rightarrow$$

Two independent components

Dirac Fermion needs independent left and right chiral projections:

$$\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_L \end{pmatrix} + \begin{pmatrix} \varphi_R \\ 0 \end{pmatrix} = \psi_L + \psi_R$$

Majorana Fermion needs only one independent chiral projection:

$$\nu = \begin{pmatrix} -i\sigma^2 \chi_L^* \\ \chi_L \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_L \end{pmatrix} + \begin{pmatrix} -i\sigma^2 \chi_L^* \\ 0 \end{pmatrix} = \nu_L + \nu_L^c$$

Two-component neutrino can have a Majorana mass!

Dirac equation (chiral representation):

$$\begin{cases} i\,(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\chi_L = m\,\chi_R \\ i\,(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\,\chi_R = m\,\chi_L \end{cases}$$

Majorana condition $\Downarrow$ $\quad \nu^c = \nu \quad \Rightarrow \quad \chi_R = -i\sigma^2\chi_L^*$

$$\left.\begin{array}{l} (\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\chi_L = -m\,\sigma^2\,\chi_L^* \\ (\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\,\sigma^2\,\chi_L^* = m\,\chi_L \end{array}\right\}\quad \text{equivalent}$$

Majorana Equation : $\quad \underbrace{(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\chi_L}_{\text{Weyl}} + m\,\sigma^2\,\chi_L^* = 0$

Two-component neutrino with Majorana mass!

> Per quanto non sia forse ancora possibile chiedere all'esperienza una decisione tra questa nuova teoria e quella consistente nella semplice estensione delle equazioni di Dirac alle particelle neutre, va tenuto presente che la prima introduce, in questo campo ancora poco esplorato, un minor numero di entità ipotetiche.
>
> E. Majorana, Nuovo Cimento 5 (1937) 171

Two-component Majorana Equation:

$$\boxed{(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\chi_L + m\,\sigma^2\,\chi_L^* = 0}$$

Four-component formalism:

$$\begin{pmatrix} 0 & i\,(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) \\ i\,(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ \chi_L \end{pmatrix}}_{\nu_L} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \underbrace{\begin{pmatrix} -i\sigma^2\chi_L^* \\ 0 \end{pmatrix}}_{\nu_L^c} = 0$$

$\Downarrow$ Chiral Representation

$$\boxed{i\gamma^\mu\partial_\mu\nu_L + m\nu_L^c = 0}$$

Lagrangian:

$$\boxed{\nu_L^c = \mathcal{C}\overline{\nu_L}^T,\ \overline{\nu_L^c} = -\nu_L^T\,\mathcal{C}^\dagger}$$

$$\mathcal{L} = \frac{1}{2}\left[-i\overline{\nu_L}\gamma^\mu(\partial_\mu\nu_L) + i(\partial_\mu\overline{\nu_L})\gamma^\mu\nu_L - m(\underbrace{\overline{\nu_L^c}\nu_L + \overline{\nu_L}\nu_L^c}_{-\nu_L^T\,\mathcal{C}^\dagger\,\nu_L\ \underbrace{+\overline{\nu_L}\,\mathcal{C}\overline{\nu_L}^T}_{-\overline{\nu_L}\,\mathcal{C}^T\,\overline{\nu_L}^T}})\right]$$

Euler-Lagrange Equations: $\partial_\mu \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\nu_L})} - \dfrac{\partial\mathcal{L}}{\partial\overline{\nu_L}} = 0$

$$\Downarrow$$

$$\frac{1}{2}(i\gamma^\mu\partial_\mu\nu_L + i\gamma^\mu\partial_\mu\nu_L + m\mathcal{C}\overline{\nu_L}^T - m\,\underbrace{\mathcal{C}^T\overline{\nu_L}^T}_{-\mathcal{C}\overline{\nu_L}^T}) = 0$$

Majorana mass term:

$$\mathcal{L}^{\mathrm{M}} = -\frac{1}{2}\,m\left(\overline{\nu_L^c}\,\nu_L + \overline{\nu_L}\,\nu_L^c\right)$$

The chiral fields ν_L and ν_R (if it exists!) are the building blocks of the neutrino Lagrangian.

Only ν_L $\Rightarrow$ Majorana mass term:

$$\mathcal{L}_{\mathrm{L}}^{\mathrm{M}} = -\frac{1}{2}\, m_L\, \overline{\nu}\, \nu = -\frac{1}{2}\, m_L \left(\overline{\nu_L} + \overline{\nu_L^c}\right) \left(\nu_L + \nu_L^c\right)$$

$$\boxed{\begin{aligned} \nu_L^c &= \mathcal{C}\overline{\nu_L}^T \\ \overline{\nu_L^c} &= -\nu_L^T \mathcal{C}^\dagger \end{aligned}}$$

$$= -\frac{1}{2}\, m_L \left(\overline{\nu_L^c}\, \nu_L + \overline{\nu_L}\, \nu_L^c\right)$$

$$= \frac{1}{2}\, m_L \,(\nu_L^T\, \mathcal{C}^\dagger\, \nu_L \underbrace{- \overline{\nu_L}\, \mathcal{C}\, \overline{\nu_L}^T}_{\nu_L^\dagger\, \mathcal{C}\, \nu_L^*})$$

ν_L and ν_R $\Rightarrow$ Dirac mass term:

$$\mathcal{L}^{\mathrm{D}} = -m_{\mathrm{D}}\, \overline{\nu}\, \nu = -\, m_{\mathrm{D}} \left(\overline{\nu_L} + \overline{\nu_R}\right) \left(\nu_L + \nu_R\right)$$

$$= -\, m_{\mathrm{D}} \left(\overline{\nu_L}\, \nu_R + \overline{\nu_R}\, \nu_L\right)$$

Surprise!: ν_L and ν_R $\Rightarrow$ Dirac-Majorana mass term:

$$\mathcal{L}^{\mathrm{D+M}} = \mathcal{L}_{\mathrm{L}}^{\mathrm{M}} + \mathcal{L}_{\mathrm{R}}^{\mathrm{M}} + \mathcal{L}^{\mathrm{D}}$$

$$= -\frac{1}{2} \begin{pmatrix} \overline{\nu_L^c} & \overline{\nu_R} \end{pmatrix} \begin{pmatrix} m_L & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \mathrm{h.c.}$$

$$= \frac{1}{2}\, N_L^T\, \mathcal{C}^\dagger\, M\, N_L + \mathrm{h.c.}$$

$$N_L = \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}, \qquad M = \begin{pmatrix} m_L & m_{\mathrm{D}} \\ m_{\mathrm{D}} & m_R \end{pmatrix}$$

$$\mathcal{L}^{\text{D+M}} = \frac{1}{2} N_L^T \, \mathcal{C}^\dagger \, M \, N_L + \text{h.c.}$$

Fields with definite mass are obtained by diagonalization:

$$N_L = U \, n_L \, , \; n_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \end{pmatrix} \Rightarrow U^T \, M \, U = \text{diag}(m_1, m_2)$$

$$\mathcal{L}^{\text{D+M}} = \frac{1}{2} \sum_{k=1,2} m_k \nu_{kL}^T \, \mathcal{C}^\dagger \, \nu_{kL} + \text{h.c.} = -\frac{1}{2} \sum_{k=1,2} m_k \overline{\nu_k} \, \nu_k$$

$$\nu_k = \nu_{kL} + \nu_{kL}^c \quad \text{Massive neutrinos are Majorana!}$$

CP invariance $\Rightarrow$ real m_L, m_R, m_{D} $\Rightarrow$ $U = \mathcal{O} \, \rho$

$$M = \begin{pmatrix} m_L & m_{\text{D}} \\ m_{\text{D}} & m_R \end{pmatrix} , \quad \mathcal{O} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} , \quad \rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}$$

$$\mathcal{O}^T M \mathcal{O} = \text{diag}(m_1', m_2') \quad \Rightarrow \quad \tan 2\vartheta = \frac{2 m_{\text{D}}}{m_R - m_L}$$

$$m_{2,1}' = \frac{1}{2} \left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4 \, m_{\text{D}}^2} \right]$$

$$m_k = \rho_k^2 \, m_k' \qquad \left\{ \begin{array}{l} \nu_{1L} = \rho_1^* \, (\cos\vartheta \, \nu_L - \sin\vartheta \, \nu_R^c) \\ \nu_{2L} = \rho_2^* \, (\sin\vartheta \, \nu_L + \cos\vartheta \, \nu_R^c) \end{array} \right.$$

$$(\rho_k^2 = \pm 1)$$

The phases ρ_1, ρ_2 give the CP parities of ν_1, ν_2 (important in neutrinoless double-β decay):

$$\nu_k(t, \vec{x}) \xrightarrow{\text{CP}} i \, \rho_k^2 \, \gamma^0 \, \nu_k(t, -\vec{x})$$

Maximal Mixing

$$m_L = m_R \Rightarrow \vartheta = \pi/4\,,\; m'_{2,1} = m_L \pm m_{\mathrm{D}}$$

$$m_{\mathrm{D}} > m_L \geq 0 \Rightarrow \begin{cases} m_1 = m_{\mathrm{D}} - m_L\,,\; \rho_1^2 = -1 \\ m_2 = m_{\mathrm{D}} + m_L\,,\; \rho_2^2 = +1 \end{cases}$$

$$\nu_{1L} = \frac{i}{\sqrt{2}}\left(\nu_L - \nu_R^c\right)\,,\; \nu_{2L} = \frac{1}{\sqrt{2}}\left(\nu_L + \nu_R^c\right)$$

Majorana neutrino fields:

$$\nu_1 = \nu_{1L} + \nu_{1L}^c = \frac{i}{\sqrt{2}}\left[\left(\nu_L - \nu_L^c\right) + \left(\nu_R - \nu_R^c\right)\right]$$

$$\nu_2 = \nu_{2L} + \nu_{2L}^c = \frac{1}{\sqrt{2}}\left[\left(\nu_L + \nu_L^c\right) + \left(\nu_R + \nu_R^c\right)\right]$$

$$m_L = m_R = 0 \Rightarrow \text{Dirac neutrino field}$$

ν_1 and ν_2 have the same mass $m_1 = m_2 = m_{\mathrm{D}}$ and opposite CP parities.

The two Majorana fields ν_1 and ν_2 can be combined to give one Dirac field ν:

$$\nu = \frac{1}{\sqrt{2}}\left(-i\nu_1 + \nu_2\right) = \nu_L + \nu_R$$

Viceversa, one Dirac field ν can always be splitted in two Majorana fields:

$$\begin{aligned}\nu &= \frac{1}{2}\left[(\nu - \nu^c) + (\nu + \nu^c)\right] \\ &= -\frac{i}{\sqrt{2}}\left(i\frac{\nu - \nu^c}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{\nu + \nu^c}{\sqrt{2}}\right) \\ &= -\frac{i}{\sqrt{2}}\,\nu_1 + \frac{1}{\sqrt{2}}\,\nu_2\end{aligned}$$

Majorana fields ($\nu_1^c = \nu_1$, $\nu_2^c = \nu_2$):

$$\nu_1 = i\frac{\nu - \nu^c}{\sqrt{2}}\,,\ \nu_2 = \frac{\nu + \nu^c}{\sqrt{2}}$$

In general: one Dirac field $\equiv$ two Majorana fields with same mass and opposite CP parities.

See-Saw Mechanism

$$m'_{2,1} = \frac{1}{2}\left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4\,m_{\mathrm{D}}^2} \right]$$

$$m_k = \rho_k^2\, m'_k \qquad (\rho_k^2 = \pm 1) \qquad\qquad \tan 2\vartheta = \frac{2m_{\mathrm{D}}}{m_R - m_L}$$

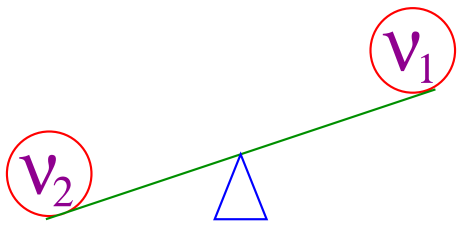
$$\boxed{m_L = 0\,,\ m_{\mathrm{D}} \ll m_R} \Rightarrow \begin{cases} m'_1 \simeq -\dfrac{(m_{\mathrm{D}})^2}{m_R} \\ m'_2 \simeq m_R \\ \tan 2\vartheta = 2\,\dfrac{m_{\mathrm{D}}}{m_R} \end{cases}$$

$$m_1 \simeq \frac{(m_{\mathrm{D}})^2}{m_R} \ll m_{\mathrm{D}}\,, \qquad \rho_1^2 = -1$$

$$m_2 \simeq m_R\,, \qquad \rho_2^2 = +1$$

$$\tan\vartheta \simeq \frac{m_{\mathrm{D}}}{m_R} \ll 1$$

See-Saw Mechanism



$$\nu_{1L} \simeq -\nu_L\,, \qquad \nu_{2L} \simeq \nu_R^c$$

What about m_L?

$$L = \underbrace{\begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}}_{\text{doublet}} \leftarrow I_3 = 1/2 \qquad , \qquad \mathcal{L}^{\mathrm{M}} \sim \underbrace{\nu_L^T \, \nu_L}_{\substack{I_3=1 \\ \text{triplet}}}$$

$$\underbrace{\frac{g}{M} \left(L^T \, L\right) \left(\phi^T \, \phi\right)}_{\text{non-renormalizable}} \xrightarrow[\text{Breaking}]{\text{Symmetry}} m_L \, \nu_L^T \, \nu_L$$

Effective Lagrangian!

$$\phi \xrightarrow[\text{Breaking}]{\text{Symmetry}} v \Rightarrow g \left(\phi^T \, \phi\right) \xrightarrow[\text{Breaking}]{\text{Symmetry}} m_{\mathrm{D}}^2 \quad \left(m_\ell^2 \text{ or } m_q^2\right)$$

$$\boxed{m_L \sim \frac{m_{\mathrm{D}}^2}{M}} \qquad \text{See-Saw Type}$$

Natural Cut-Off: $\qquad M_{\mathrm{P}} \sim 10^{19} \, \mathrm{GeV}$

Even in the SM it is natural to expect small Majorana ν masses given by the see-saw type relation

$$\boxed{m_L \sim \frac{m_{\mathrm{D}}^2}{M_{\mathrm{P}}}}$$

General consideration on fermion masses

In SM fermion masses are generated through Yukawa couplings:

$$\mathcal{L}_{\mathrm{Y}} = \sum_{\ell,\ell'} y_{\ell,\ell'} \, \overline{L_\ell} \, \phi \, \ell'_R$$

The coefficients $y_{\ell,\ell'}$ are parameters of the model

$\Downarrow$

Their explanation must come from new physics BSM

$\Downarrow$

All fermion masses give info on new physics BSM

Smallness of ν masses is additional mistery

$\Downarrow$

More info on new physics BSM

Known natural explanations of smallness of ν masses:

⋆ See-Saw Mechanism

⋆ Effective Lagrangian

$$\text{Both imply} \left\{ \begin{array}{l} \text{Majorana } \nu \text{ masses!} \\ \boxed{m_{\text{light}} \sim \dfrac{m_{\mathrm{D}}^2}{M}} \end{array} \right.$$

Dirac and Majorana Degrees of Freedom

Parity (Space Inversion): $t \xrightarrow{\text{P}} t \,, \quad \vec{x} \xrightarrow{\text{P}} -\vec{x}$

$$\vec{p} \xrightarrow{\text{P}} -\vec{p}\,, \qquad \vec{L} = \vec{x} \times \vec{p} \xrightarrow{\text{P}} \vec{L} \quad \Rightarrow \quad \vec{s} \xrightarrow{\text{P}} \vec{s}$$

Helicity: $h = \dfrac{\vec{s} \cdot \vec{p}}{|\vec{p}|} \xrightarrow{\text{P}} -h$

Time reversal: $t \xrightarrow{\text{T}} -t \,, \quad \vec{x} \xrightarrow{\text{T}} \vec{x}$

$$\vec{p} \xrightarrow{\text{T}} -\vec{p}\,, \quad \vec{L} = \vec{x} \times \vec{p} \xrightarrow{\text{T}} -\vec{L} \quad \Rightarrow \quad \vec{s} \xrightarrow{\text{T}} -\vec{s}$$

Helicity: $h = \dfrac{\vec{s} \cdot \vec{p}}{|\vec{p}|} \xrightarrow{\text{T}} h$

Space-Time Inversion: $t \xrightarrow{\text{PT}} -t \,, \quad \vec{x} \xrightarrow{\text{PT}} -\vec{x}$

$$\vec{p} \xrightarrow{\text{PT}} \vec{p}\,, \qquad h \xrightarrow{\text{PT}} -h$$

$$\nu(\vec{p}, h) \xrightarrow{\text{PT}} \nu(\vec{p}, -h)$$

CPT: $\begin{cases} \nu(\vec{p}, h) \xrightarrow{\text{CPT}} \bar{\nu}(\vec{p}, -h) & \text{Dirac} \\ \nu(\vec{p}, h) \xrightarrow{\text{CPT}} \nu(\vec{p}, -h) & \text{Majorana} \end{cases}$

Dirac Neutrino

$$
\begin{tikzcd}[column sep=large, row sep=large]
\boxed{\nu(\vec{p},h)} \arrow[r, Leftrightarrow, "\text{CPT}"] \arrow[d, Leftrightarrow, "\text{Boost}"'] & \boxed{\bar{\nu}(\vec{p},-h)} \arrow[d, Leftrightarrow, "\text{Boost}"] \\
\boxed{\nu(-\vec{p},-h)} \arrow[r, Leftrightarrow, "\text{CPT}"'] & \boxed{\bar{\nu}(-\vec{p},h)}
\end{tikzcd}
$$

$$
\left.\begin{array}{r}
\nu(\vec{p},h) \text{ and } \bar{\nu}(-\vec{p},h) \\
\nu(-\vec{p},-h) \text{ and } \bar{\nu}(\vec{p},-h)
\end{array}\right\} \text{have different interactions}
$$

$$\Downarrow$$

four states

Majorana Neutrino

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\nu(\vec{p},h)} \arrow[r, Leftrightarrow, "\text{CPT}"] \arrow[d, Leftrightarrow, "\text{Boost}"'] \arrow[dr, Leftrightarrow] \arrow[dr, phantom, "180^\circ\ \text{Rotation}"] & \boxed{\nu(\vec{p},-h)} \arrow[d, Leftrightarrow, "\text{Boost}"] \arrow[dl, Leftrightarrow] \\
\boxed{\nu(-\vec{p},-h)} \arrow[r, Leftrightarrow, "\text{CPT}"'] & \boxed{\nu(-\vec{p},h)}
\end{tikzcd}
$$

$$
\left.\begin{array}{r}
\nu(\vec{p},h) \text{ and } \nu(-\vec{p},h) \\
\nu(-\vec{p},-h) \text{ and } \nu(\vec{p},-h)
\end{array}\right\} \text{have same interactions}
$$

$$\Downarrow$$

two states


$\nu(-\vec{p}, h=-1)$
$\vec{s}$
$\vec{p}$
$\nu(\vec{p}, h=-1)$

Majorana Neutrino $\Rightarrow$ No conserved Lepton Number

$L_e\,,\ L_\mu\,,\ L_\tau\,,\ L = L_e + L_\mu + L_\tau$

$$\cancel{L = -1} \quad \longleftarrow \quad \nu^c = \nu \quad \longrightarrow \quad \cancel{L = +1}$$

$$\text{conserved Lepton Number} \quad \underset{\text{Theorem}}{\overset{\text{Noether}}{\rightleftarrows}} \quad \text{Global Gauge Invariance}$$

Dirac mass term: $\mathcal{L}^{\mathrm{D}} = -m_{\mathrm{D}}\left(\overline{\nu_L}\,\nu_R + \overline{\nu_R}\,\nu_L\right)$

invariant under

$$\nu_L \to e^{i\Lambda}\nu_L\,, \qquad \nu_R \to e^{i\Lambda}\nu_R$$

$$\overline{\nu_R} \to e^{-i\Lambda}\overline{\nu_R}\,, \qquad \overline{\nu_L} \to e^{-i\Lambda}\overline{\nu_L}$$

Majorana mass term: $\mathcal{L}_{\mathrm{M}} = -m_{\mathrm{M}}\left(\overline{\nu_L}\,\nu_L^c + \overline{\nu_L^c}\,\nu_L\right)$

not invariant under

$$\nu_L \to e^{i\Lambda}\nu_L\,, \qquad \overline{\nu_L} \to e^{-i\Lambda}\overline{\nu_L}$$

$$\overline{\nu_L^c} \to e^{i\Lambda}\overline{\nu_L^c}\,, \qquad \nu_L^c \to e^{-i\Lambda}\nu_L^c$$

Majorana Neutrino $=$ Truly Neutral Fermion

... in 1937 only the notion of electric charge was known. Now Majorana, *first*, invented explicitly truly neutral fermions or Majorana particles, that is fermions which are identical with their own antiparticle.

Second, Majorana, putting the question about an electrically neutral fermion being described either by his theory or by the Dirac theory, implicitly introduced the notion of charges other than electrical. Majorana particles particles are fermions which have neither electrical nor any other charges. Electrically neutral fermions which are not Majorana particles are described by the Dirac theory, are not truly neutral and have a (non-electric) charge. Notice that explicitly the notion of baryon and lepton charges were introduced only in 1949 and 1953!

B. Pontecorvo, J. de Physique 12 (1982) C8

Neutrino Mass

Direct measurements of neutrino masses $\Rightarrow$ upper limits:

$$m_{\nu_e} \lesssim 3\,\text{eV} \qquad (^3\text{H} \to {}^3\text{He} + e^- + \bar{\nu}_e)$$

$$m_{\nu_\mu} < 190\,\text{keV}\ (90\%\ \text{CL}) \qquad (\pi^- \to \mu^- + \bar{\nu}_\mu)$$

$$m_{\nu_\tau} < 18.2\,\text{MeV}\ (95\%\ \text{CL}) \qquad (\tau^- \to 2\pi^+ + 3\pi^- + \nu_\tau)$$

Cosmological limit on neutrino masses:

$$\sum_{\text{light }\nu} m_\nu \lesssim 24\,\text{eV} \qquad (n_\nu \sim 300\,\text{cm}^{-3})$$