Neutrino Mixing

Minimally extended SM $\Rightarrow$ Add ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$

$\Downarrow$

Dirac neutrino mass term can be generated by the standard Higgs mechanism

$$\mathcal{L}^{\mathrm{D}} = -\sum_{\ell,\ell'} \overline{\nu_{\ell R}}\, M^{\mathrm{D}}_{\ell\ell'}\, \nu_{\ell' L} + \text{h.c.} \qquad (\ell = e, \mu, \tau)$$

$$M^{\mathrm{D}} = \text{complex } 3 \times 3 \text{ matrix}$$

Diagonalization: biunitary transformation

$$V^{\dagger}\, M^{\mathrm{D}}\, U = M\,, \qquad M_{kj} = m_k\, \delta_{kj}\,, \qquad m_k \geq 0$$

$$V^{\dagger} = V^{-1}\,, \qquad U^{\dagger} = U^{-1}$$

Possible?

Proof that diagonalization of M^{D} by biunitary transformation is possible

Consider $M^{\mathrm{D}} M^{\mathrm{D}\dagger}$:

Hermitian $\Rightarrow$ diagonalized by unitary transformation

$$V^\dagger M^{\mathrm{D}} M^{\mathrm{D}\dagger} V = M^2 \,, \qquad M^2_{kj} = m_k^2 \, \delta_{kj}$$

Choosing an appropriate matrix U, it is always possible to write

$$M^{\mathrm{D}} = V \, M \, U^\dagger \,, \qquad M_{kj} = \sqrt{m_k^2} \, \delta_{kj}$$

$$U^\dagger = M^{-1} \, V^\dagger \, M^{\mathrm{D}} \,, \qquad U = M^{\mathrm{D}\dagger} \, V \, M^{-1} \qquad (M^\dagger = M)$$

Magically U is unitary!

$$U^\dagger U = M^{-1} V^\dagger M^{\mathrm{D}} M^{\mathrm{D}\dagger} V M^{-1} = 1$$

$$\begin{aligned} U U^\dagger &= M^{\mathrm{D}\dagger} V M^{-2} V^\dagger M^{\mathrm{D}} \\ &= M^{\mathrm{D}\dagger} V V^\dagger (M^{\mathrm{D}\dagger})^{-1} (M^{\mathrm{D}})^{-1} V V^\dagger M^{\mathrm{D}} = 1 \end{aligned}$$

Then $m_k = \sqrt{m_k^2} \geq 0$

Diagonalized Dirac mass term:

$$\mathcal{L}^{\mathrm{D}} = - \sum_{k=1}^{3} m_k \, \overline{\nu_k} \, \nu_k$$

Mixing:

$$\left. \begin{aligned} \nu_{\ell L} &= \sum_{k=1}^{3} U_{\ell k} \, \nu_{kL} \\ \nu_{\ell R} &= \sum_{k=1}^{3} V_{\ell k} \, \nu_{kR} \end{aligned} \right\} \qquad (\ell = e, \mu, \tau)$$

No right-handed fields in weak interaction Lagrangian

$$\Downarrow$$

Right-handed singlets are sterile and not mixed with active neutrinos.

Lepton weak charged current:

$$j_\rho^{\mathrm{CC}\,\dagger} = 2 \sum_{\ell=e,\mu,\tau} \overline{\ell_L} \, \gamma_\rho \, \nu_{\ell L} = 2 \sum_{\ell=e,\mu,\tau} \sum_{k=1}^{3} \overline{\ell_L} \, \gamma_\rho \, U_{\ell k} \, \nu_{kL}$$

Unitary 3×3 mixing matrix U

$\Downarrow$

3 mixing angles + 6 phases

Only one measurable phase

$\Downarrow$

3 mixing angles + 1 physical CP-violating phase

Convenient parameterization:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

$$c_{ij} \equiv \cos\vartheta_{ij} \,, \qquad s_{ij} \equiv \sin\vartheta_{ij}$$

In this parameterization δ_{13} is associated with s_{13} $\Rightarrow$ obviously CP violation is small if ϑ_{13} is small.

In other parameterizations the phase can be associated with s_{12} or s_{23} $\Rightarrow$ CP violation is small if any mixing angle is small.

If any of the elements of the mixing matrix is zero, the phase can be rotated away $\Rightarrow$ no CP violation.

PHYSICAL PARAMETERS IN N×N MIXING MATRIX

UNITARY N×N MATRIX $\Rightarrow N^2$ PARAMETERS

$\frac{N(N-1)}{2}$ MIXING ANGLES

$\frac{N(N+1)}{2}$ PHASES

WEAK CHARGED CURRENT

$$j_\rho^{cc\dagger} = 2 \sum_{\ell} \bar{\ell}_L \gamma_\rho \nu_{\ell L}$$

$$= 2 \sum_{\ell, k} \bar{\ell}_L \gamma_\rho U_{\ell k} \nu_{kL}$$

LAGRANGIAN IS INVARIANT UNDER GLOBAL PHASE TRANSFORMATIONS OF DIRAC FIELDS (CONSERVATION OF L, B)

$$\ell_L \rightarrow e^{i\vartheta_\ell} \ell_L \qquad \nu_{kL} \rightarrow e^{i\varphi_k} \nu_{kL}$$

$$j_\rho^{cc\,\dagger} \rightarrow 2 \sum_{k,\ell} \bar{\ell}_L \gamma_\rho \, e^{-i\theta_\ell} U_{\ell k} \, e^{i\varphi_k} \nu_{kL}$$

$$= 2 \sum_{k,\ell} \bar{\ell}_L \gamma_\rho \, \underset{\substack{\uparrow \\ 1}}{e^{-i(\theta_e - \varphi_1)}} \, \underset{\substack{\uparrow \\ N-1}}{e^{-i(\theta_\ell - \theta_e)}} U_{\ell k} \, \underset{\substack{\uparrow \\ N-1}}{e^{i(\varphi_k - \varphi_1)}} \nu_{kL}$$

NUMBER OF INDEPENDENT PHASES THAT CAN BE ELIMINATED:

$$1 + (N-1) + (N-1) = 2N - 1 \quad (\text{NOT } 2N!)$$

NUMBER OF PHYSICAL PHASES:

$$\frac{N(N+1)}{2} - (2N-1) = \frac{(N-1)(N-2)}{2}$$

$$N = 3 \Rightarrow \begin{cases} 3 \text{ MIXING ANGLES} \\ 1 \text{ PHYSICAL PHASE} \end{cases}$$

DIRAC ν MASS TERM ALLOWS

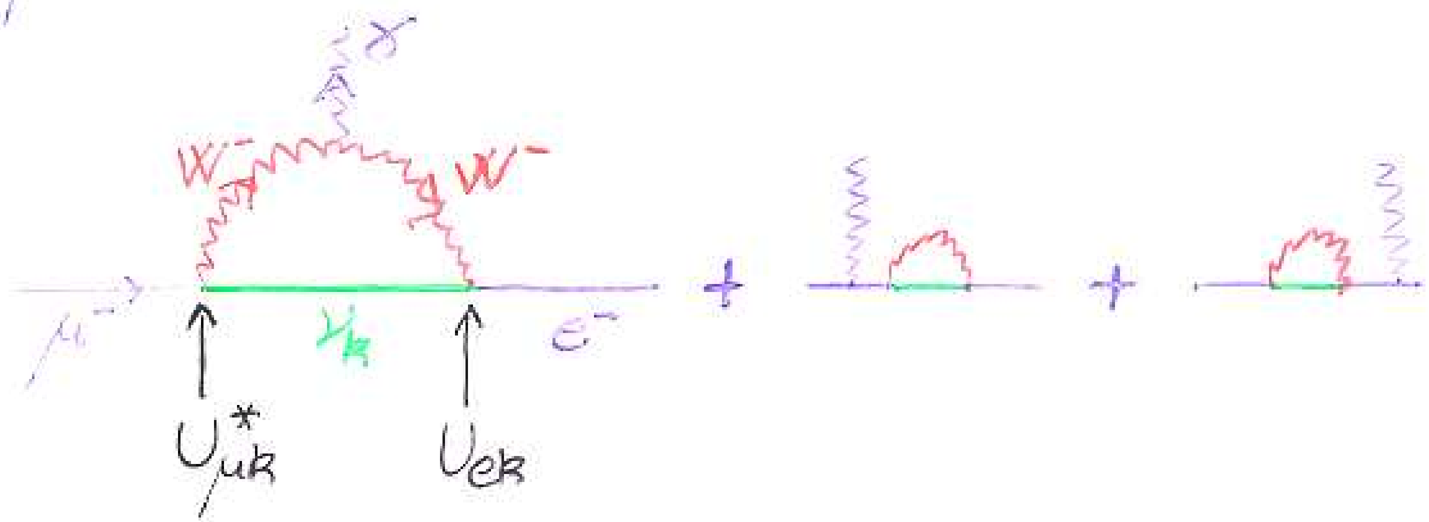
L_e, L_μ, L_τ VIOLATING PROCESSES

LIKE:

$$\mu^\pm \to e^\pm + \gamma$$

$$\mu^\pm \to e^\pm + e^+ + e^-$$

$\mu^- \to e^- + \gamma$



$$\sum_k U^*_{\mu k} U_{ek} = 0 \Rightarrow \text{GIM MECHANISM}$$

$$\Gamma = \frac{G_F m_\mu^5}{192\pi^3} \underbrace{\frac{3}{32}\,\frac{\alpha}{\pi} \left| \sum_k U^*_{\mu k} U_{ek} \frac{m_k^2}{m_W^2} \right|^2}_{\text{BR}}$$

SUPPRESSION FACTOR

$m_k \lesssim 10^2\,\text{eV} \Rightarrow \dfrac{m_k^2}{m_W^2} \lesssim 10^{-18}$

$\text{BR} \lesssim 10^{-41}$ $\quad (\text{BR})_{\text{EXP}} \lesssim 10^{-11}$

30 ORDERS OF MAGNITUDE SMALLER!

Dirac-Majorana mass term

active $\nu_{\ell L}$ $(\ell = e, \mu, \tau)$

$+$

sterile ν_{sR} $(s = s_1, s_2, \ldots, s_{\mathcal{N}})$

$$\mathcal{L}^{\mathrm{D+M}} = \mathcal{L}_L^{\mathrm{M}} + \mathcal{L}^{\mathrm{D}} + \mathcal{L}_R^{\mathrm{M}}$$

$$\mathcal{L}^{\mathrm{D}} = -\sum_{s,\ell} \overline{\nu_{sR}}\, M_{s\ell}^{\mathrm{D}}\, \nu_{\ell L} + \mathrm{h.c.}$$

$$\mathcal{L}_L^{\mathrm{M}} = -\frac{1}{2} \sum_{\ell,\ell'} \overline{\nu_{\ell L}^{c}}\, M_{\ell\ell'}^{L}\, \nu_{\ell' L} + \mathrm{h.c.}$$

$$\mathcal{L}_R^{\mathrm{M}} = -\frac{1}{2} \sum_{s,s'} \overline{\nu_{sR}}\, M_{ss'}^{R}\, \nu_{s' R}^{c} + \mathrm{h.c.}$$

M^{D}, M^L, M^R are complex matrices

M^L, M^R are symmetric

For example: $\boxed{\nu^c_{\ell L} = \mathcal{C}\,\overline{\nu_{\ell L}}^T\,,\ \overline{\nu^c_{\ell L}} = -\nu^T_{\ell L}\,\mathcal{C}^\dagger}$

$$\sum_{\ell,\ell'} \overline{\nu^c_{\ell L}}\, M^L_{\ell\ell'}\, \nu_{\ell' L} = -\sum_{\ell,\ell'} \nu^T_{\ell' L}\, \mathcal{C}^{-1}\, M^L_{\ell\ell'}\, \nu_{\ell L}$$

$$\boxed{\mathcal{C}^T = -\mathcal{C}} \rightarrow \ = -\sum_{\ell,\ell'} \nu^T_{\ell L}\, \mathcal{C}^{-1}\, M^L_{\ell\ell'}\, \nu_{\ell' L}$$

$$\boxed{\ell \leftrightarrows \ell'} \rightarrow \ = \sum_{\ell',\ell} \overline{\nu^c_{\ell' L}}\, M^L_{\ell'\ell}\, \nu_{\ell L}$$

$$\Downarrow$$

$M^L_{\ell\ell'} = M^L_{\ell'\ell} \Leftrightarrow M^L$ is symmetric!

Column matrix of all left-handed fields:

$$N_L \equiv \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} , \quad \nu_L \equiv \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} , \quad \nu_R^c \equiv \begin{pmatrix} \nu_{s_1 R}^c \\ \vdots \\ \nu_{s_{\mathcal{N}} R}^c \end{pmatrix}$$

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} \overline{N_L^c} \, M^{\text{D+M}} \, N_L + \text{h.c.}$$

$(3+\mathcal{N}) \times (3+\mathcal{N})$ symmetric mixing matrix:

$$M^{\text{D+M}} \equiv \begin{pmatrix} M^L & (M^{\text{D}})^T \\ M^{\text{D}} & M^R \end{pmatrix}$$

Diagonalized by the transformation $(U^\dagger = U^{-1})$

$$U^T \, M^{\text{D+M}} \, U = M \, , \quad M_{kj} = m_k \, \delta_{kj} \, , \quad m_k \geq 0$$

Fields with definite mass:

$$n_L \equiv \begin{pmatrix} \nu_{1L} \\ \vdots \\ \nu_{(3+\mathcal{N})L} \end{pmatrix} = U^\dagger \, N_L$$

$$\mathcal{L}^{\text{D+M}} = -\frac{1}{2} \overline{n_L^c} \, M \, n_L + \text{h.c.} = -\frac{1}{2} \sum_{k=1}^{3+\mathcal{N}} m_k \, \overline{\nu_{kL}^c} \, \nu_{kL} + \text{h.c.}$$

Fields with definite mass are Majorana:

$$n \equiv \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_{3+\mathcal{N}} \end{pmatrix} = n_L + n_L^c = U^\dagger \, N_L + U^T \, N_L^c$$

$$\mathcal{L}^{\mathrm{D+M}} = -\frac{1}{2} \, \overline{n} \, M \, n = -\frac{1}{2} \sum_{k=1}^{3+\mathcal{N}} m_k \, \overline{\nu_k} \, \nu_k + \text{h.c.}$$

Mixing relations:

$$\nu_{\ell L} = \sum_{k=1}^{3+\mathcal{N}} U_{\ell k} \, \nu_{kL} \qquad (\ell = e, \mu, \tau)$$

$$\nu_{sR}^c = \sum_{k=1}^{3+\mathcal{N}} U_{sk} \, \nu_{kL} \qquad (s = s_1, \ldots, s_{\mathcal{N}})$$

Sterile neutrino fields ν_{sR} are connected to Active neutrino fields $\nu_{\ell L}$ trough the Massive neutrino fields ν_{kL}

$$\Downarrow$$

Active $\leftrightarrows$ Sterile oscillations are possible!

Lepton weak charged current:

$$j_\rho^{\mathrm{CC}\dagger} = 2 \sum_{\ell=e,\mu,\tau} \overline{\ell_L} \, \gamma_\rho \, \nu_{\ell L} = 2 \sum_{\ell=e,\mu,\tau} \sum_{k=1}^{3+\mathcal{N}} \overline{\ell_L} \, \gamma_\rho \, U_{\ell k} \, \nu_{kL}$$

PHYSICAL PARAMETERS IN $N\times N$ MIXING MATRIX FOR MAJORANA NEUTRINOS

$$N^2 \begin{cases} \dfrac{N(N-1)}{2} & \text{MIXING ANGLES} \\[2ex] \dfrac{N(N+1)}{2} & \text{PHASES} \end{cases}$$

WEAK CHARGED CURRENT:

$$j_\rho^{cc\dagger} = 2 \sum_{\ell,k} \underset{\substack{\uparrow \\ \text{CAN BE} \\ \text{REPHASED}}}{\overline{\ell_L}} \gamma_\rho U_{\ell k} \underset{\substack{\uparrow \\ \text{CANNOT BE} \\ \text{REPHASED}}}{\nu_{kL}}$$

MAJORANA MASS TERM $\sim \nu_{kL}^T \nu_{kL} + h.c.$

IS NOT INVARIANT UNDER REPHASING!

$$\nu_{kL} \to e^{i\varphi_k} \nu_{kL} \implies \nu_{kL}^T \nu_{kL} \to e^{2i\varphi_k} \nu_{kL}^T \nu_{kL}$$

GLOBAL LEPTON NUMBER IS NOT CONSERVED!

ONLY N PHASES CORRESPONDING TO CHARGED LEPTON FIELDS CAN BE ELIMINATED!

NUMBER OF PHYSICAL PHASES:

$$\frac{N(N+1)}{2} - N = \frac{N(N-1)}{2}$$

SAME NUMBER AS MIXING ANGLES

$$\frac{N(N-1)}{2} = \underbrace{\frac{(N-1)(N-2)}{2}}_{\text{"NORMAL" DIRAC PHASES}} + \underbrace{N-1}_{\text{MAJORANA PHASES}}$$

$N=3 \Rightarrow$
- 3 MIXING ANGLES
- 1 DIRAC PHASE
- 2 MAJORANA PHASES

$$U^{(M)}_{ek} = U^{(D)}_{ek} \, e^{i\lambda_k}$$

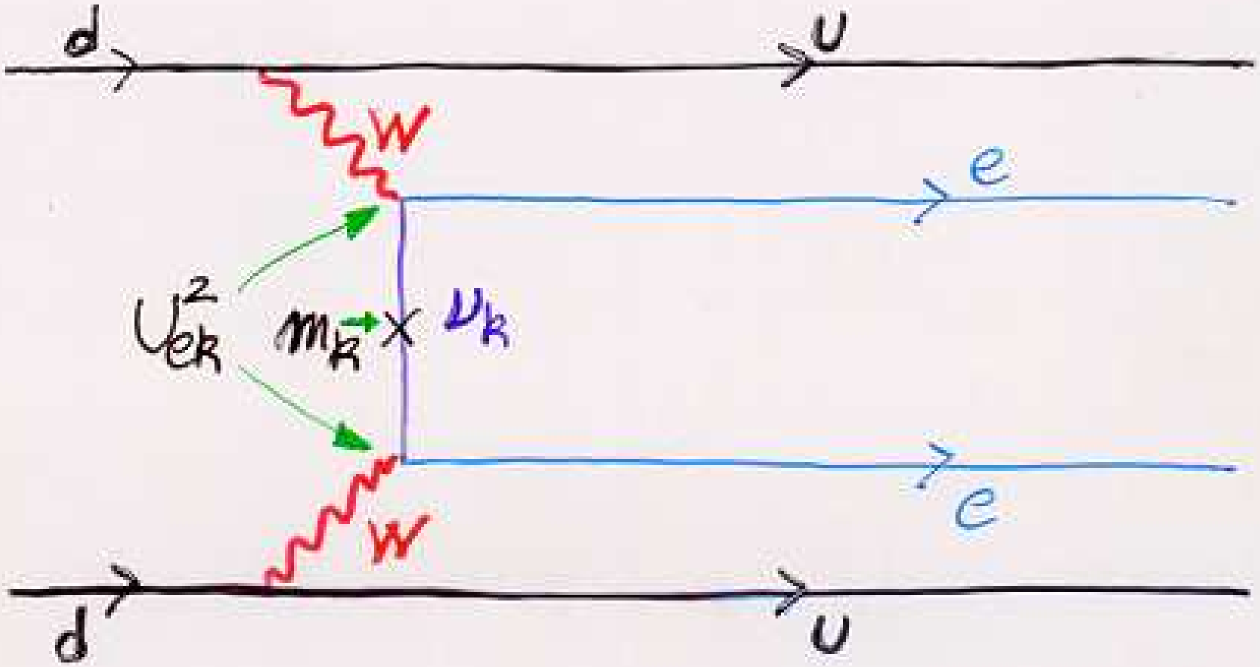
1 DIRAC PHASE

2 MAJORANA PHASES ($\lambda_1 = 0$)

NEUTRINOLESS $\beta\beta$

EFFECTIVE MAJORANA MASS

$$|\langle m \rangle| = \left| \sum_k U_{ek}^2 \, m_k \right| = |m_{ee}|$$



EXPERIMENTAL UPPER BOUND:

$$|\langle m \rangle| \lesssim 0.2 - 0.4 \text{ eV}$$

HEIDELBERG-MOSCOW
^{76}Ge

MAJORANA PHASES ARE IMPORTANT!

CAN LEAD TO CANCELLATIONS!

EXAMPLE: 2 MASSIVE NEUTRINOS

$$|\langle m \rangle| = \left| U_{e1}^2 m_1 + U_{e2}^2 m_2 \right|$$

$$U_{e1} = |U_{e1}| \qquad U_{e2} = |U_{e2}| e^{i\lambda_2}$$

$$|\langle m \rangle| = \left| |U_{e1}|^2 m_1 + |U_{e2}|^2 e^{2i\lambda_2} m_2 \right|$$

$$\lambda_2 = \pi/2 \Rightarrow |\langle m \rangle| = \left| |U_{e1}|^2 m_1 - |U_{e2}|^2 m_2 \right|$$

NO CP VIOLATION
(OPPOSITE CP PARITIES)

CANCELLATION

THIS IS WHAT HAPPENS FOR A DIRAC $\nu \equiv$ 2 MAJORANA WITH EQUAL MASS AND OPPOSITE CP PARITIES

$$\left.\begin{array}{l} |U_{e1}|^2 = |U_{e2}|^2 = \frac{1}{2} \\ m_1 = m_2 \end{array}\right\} \Rightarrow \langle m \rangle = 0$$

See-Saw Mechanism

$$M_L = 0 \Rightarrow M^{\mathrm{D+M}} = \begin{pmatrix} 0 & (M^{\mathrm{D}})^T \\ M^{\mathrm{D}} & M^R \end{pmatrix}$$

If the eigenvalues of M^R are much bigger than the elements of M^{D}, $M^{\mathrm{D+M}}$ is block-diagonalized (up to corrections of order $(M^R)^{-1}M^{\mathrm{D}}$) by the transformation ($W^\dagger \simeq W^{-1}$)

$$W^T\, M^{\mathrm{D+M}}\, W \simeq \begin{pmatrix} M_{\mathrm{light}} & 0 \\ 0 & M_{\mathrm{heavy}} \end{pmatrix}$$

$$W = 1 - \frac{1}{2}\begin{pmatrix} (M^{\mathrm{D}})^\dagger (M^R (M^R)^\dagger)^{-1} M^{\mathrm{D}} & 2(M^{\mathrm{D}})^\dagger (M^R)^{\dagger\,-1} \\ -2(M^R)^{-1} M^{\mathrm{D}} & (M^R)^{-1} M^{\mathrm{D}} (M^{\mathrm{D}})^\dagger (M^R)^{\dagger\,-1} \end{pmatrix}$$

$$M_{\mathrm{light}} \simeq -(M^{\mathrm{D}})^T\,(M^R)^{-1}\,M^{\mathrm{D}}\,, \qquad M_{\mathrm{heavy}} \simeq M^R$$

$M^R = \mathcal{M}\, I \;\Rightarrow\;$ quadratic see-saw

$$M_{\mathrm{light}} \simeq -\frac{(M^{\mathrm{D}})^T\, M^{\mathrm{D}}}{\mathcal{M}}\,, \qquad m_k \sim \frac{(m_k^f)^2}{\mathcal{M}}\ (k = 1, 2, 3)$$

$$m_1 : m_2 : m_3 = (m_1^f)^2 : (m_2^f)^2 : (m_3^f)^2$$

$M^R = \frac{\mathcal{M}}{\mathcal{M}_{\mathrm{D}}}\, M_{\mathrm{D}}$ ($\mathcal{M}_{\mathrm{D}} \sim$ scale of M_{D}) $\Rightarrow$ linear see-saw

$$M_{\mathrm{light}} \simeq -\frac{\mathcal{M}_{\mathrm{D}}}{\mathcal{M}}\, M^{\mathrm{D}}\,, \qquad m_k = \frac{\mathcal{M}_{\mathrm{D}}}{\mathcal{M}}\, m_k^f\ (k = 1, 2, 3)$$

$$m_1 : m_2 : m_3 = m_1^f : m_2^f : m_3^f$$

NEUTRINO OSCILLATIONS

[BILENKY & PONTECORVO, PHYS. REP. 41 (1978) 225]

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} W^\rho \sum_{\alpha=e,\mu,\tau} \overline{\alpha_L} \gamma_\rho \nu_{\alpha L} + \text{h.c.}$$

$$\nu_{\alpha L} = \sum_k U_{\alpha k} \nu_{kL} \longleftarrow m_k$$

NEUTRINO STATES:

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k}^* |\nu_k\rangle$$

TIME EVOLUTION: $|\nu_k(t)\rangle = e^{-iE_k t} |\nu_k\rangle$

$$|\nu_\alpha(t)\rangle = \sum_k U_{\alpha k}^* e^{-iE_k t} |\nu_k\rangle$$

$$|\nu_k\rangle = \sum_\beta U_{\beta k} |\nu_\beta\rangle$$

$$|\nu_\alpha(t)\rangle = \sum_\beta \underbrace{\left(\sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right)}_{\mathcal{A}_{\alpha\beta}(t)} |\nu_\beta\rangle$$

TRANSITION PROBABILITY:

$$P_{\nu_\alpha \to \nu_\beta}(t) = \left| \langle \nu_\beta | \nu_\alpha(t) \rangle \right|^2 = \left| \mathcal{A}_{\alpha\beta}(t) \right|^2$$

$$P_{\nu_\alpha \to \nu_\beta}(t) = \left| \sum_k U_{\alpha k}^* e^{-iE_k t} U_{\beta k} \right|^2$$

$$= \left| \cancel{e^{-iE_1 t}} \sum_k U_{\alpha k}^* e^{-i(E_k - E_1)t} U_{\beta k} \right|^2$$

THE TRANSITION PROB. DEPEND ONLY ON THE ENERGY DIFFERENCES $E_k - E_1$

RELATIVISTIC APPROXIMATION:

$$E_k \equiv \sqrt{p^2 + m_k^2} \simeq p + \frac{m_k^2}{2p}$$

$$\boxed{\begin{array}{c}\text{ASSUMPTION} \\ p_k = p_j = p\end{array}}$$

$$E_k - E_1 \simeq \frac{\Delta m_{k1}^2}{2p}$$

$\simeq E$

$$\boxed{\Delta m_{kj}^2 \equiv m_k^2 - m_j^2}$$

$$(E_k - E_1)\, t \simeq \frac{\Delta m_{k1}^2 L}{2E}$$

$\simeq L$

$$\boxed{P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* U_{\beta k}\, e^{-i \frac{\Delta m_{k1}^2 L}{2E}} \right|^2}$$

$$\boxed{P_{\nu_\alpha \to \nu_\beta}(L) = \left| \sum_k U_{\alpha k}^* U_{\beta k}\, e^{-i \frac{\Delta m_{k1}^2 L}{2E}} \right|^2}$$

TWO GENERATIONS: $k = 1, 2$

$$U = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \qquad \Delta m^2 \equiv \Delta m_{21}^2 \equiv m_2^2 - m_1^2$$

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sin^2 2\vartheta \, \sin^2 \frac{\Delta m^2 L}{4E} \qquad (\beta \neq \alpha)$$

$$P_{\nu_\alpha \to \nu_\alpha}(L) = 1 - \sin^2 2\vartheta \, \sin^2 \frac{\Delta m^2 L}{4E}$$

GENERAL FORMULA FOR ANY NUMBER OF GENERATIONS: $k, j = 1, 2, \ldots$

$$P_{\nu_\alpha \to \nu_\beta} = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 + 2\,\mathrm{Re} \sum_{k > j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E}\right)$$

DIRAC = MAJORANA
IN ν OSCILLATIONS

DIRAC: $U^{(D)}_{\alpha k}$

MAJORANA: $U^{(M)}_{\alpha k} = U^{D}_{\alpha k}\, e^{i\lambda_k}$

$$P^{(\mathrm{MAJORANA})}_{\nu_\alpha \to \nu_\beta} = \left| \sum_k U^{(M)*}_{\alpha k} U^{(M)}_{\beta k} \exp\left(-i \frac{\Delta m^2_{kj} L}{2E}\right) \right|^2$$

$$= \left| \sum_k U^{(D)*}_{\alpha k} \cancel{e^{-i\lambda_k}}\, U^{(D)}_{\beta k} \cancel{e^{i\lambda_k}} \exp\left(-i \frac{\Delta m^2_{kj} L}{2E}\right) \right|^2$$

$$= P^{(\mathrm{DIRAC})}_{\nu_\alpha \to \nu_\beta}$$

IN MATTER: SCHRÖDINGER EQUATION

$$i \frac{d \psi_\alpha}{dt} = \left(\frac{M^2_{\alpha\beta}}{2E} + V_\alpha \delta_{\alpha\beta} \right) \psi_\beta$$

$$= \left(U_{\alpha k} \frac{m_k^2}{2E} U^*_{\beta k} + V_\alpha \delta_{\alpha\beta} \right) \psi_\beta$$

MAJORANA: $\cancel{e^{i\lambda_k}}$ $\cancel{e^{-i\lambda_k}}$