MATTER EFFECTS in NEUTRINO PROPAGATION

In many (most) circumstances, neutrinos propagate in a medium filled with material.

For example solar neutrinos are created in the center of the sun, and have to travel across a lot of solar material, many of the atmospheric neutrinos have to cross a thick layer, or even the entire Earth before reaching the detector, even reactors and accelerator neutrinos, can have to cross thick layers of material.

The presence of matter can have a profound effects on the oscillation probabilities.

Neutrino Effective Potential in matter

Neutrinos have an extremely feeble interaction with matter, and therefore one could expect that the presence of matter has a negligible effect on their propagation, however this is not so because one must consider the *coherent* effect of forward scattering from many particles on the propagation of a neutrino. The effect of coherent forward scattering can be described as an index of refraction for neutrinos in matter, or equivalently as the presence of an
Effective potential

In Ordinary matter, (composed of e, p, n) the total effective potential is the sum of the three contributions. For example for ν_e:

$$V_{\nu_e} = V_{\nu_e e} + V_{\nu_e p} + V_{\nu_e n}$$

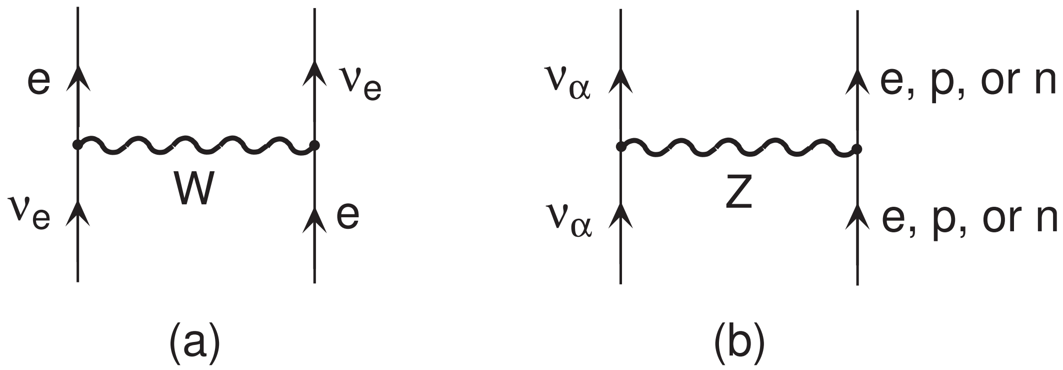
Contribution of Z_0 exchange:

$$V_{\nu_\mu e} = V_{\nu_\tau e} = \qquad -\frac{\sqrt{2}}{2}\,G_F\,N_e$$

$$V_{\nu_\mu p} = V_{\nu_\tau p} = V_{\nu_e p} = +\frac{\sqrt{2}}{2}\,G_F\,N_p$$

$$V_{\nu_\mu n} = V_{\nu_\tau n} = V_{\nu_e n} = -\frac{\sqrt{2}}{2}\,G_F\,N_e$$

The effective potential for ν_e is different because the scattering can also proceed with W exchange:



$$V_{\nu_e e} = V^W_{\nu_e e} + V^Z_{\nu_e e} = -\frac{\sqrt{2}}{2}\,G_F\,N_e + \sqrt{2}\,G_F\,N_e$$

Combining terms
for neutral matter ($N_e = N_p$):

$$V_{\nu_e} = -\frac{\sqrt{2}}{2}\, G_F N_n + \sqrt{2} G_F\, N_e$$

$$V_{\nu_\tau} = -\frac{\sqrt{2}}{2}\, G_F\, N_n$$

$$V_{\nu_\tau} = -\frac{\sqrt{2}}{2}\, G_F\, N_n$$

For anti–neutrinos
the effective potential reverse it sign:

$$V_{\overline{\nu}_\alpha} = -V_{\nu_\alpha}$$

What is relevant for Netrino propagation is the *difference of potential* between the ν_e and $\nu_{\mu,\tau}$:

$$V \equiv V_{\nu_e} - V_{\nu_\mu} = V_{\nu_e} - V_{\nu_\tau} = +\sqrt{2}\, G_F\, N_e$$

The effective potential is proportional to the electron density N_e, and has a numerical value:

$$\begin{aligned} V &= +\sqrt{2}\, G_F\, N_e \\ &= +\sqrt{2}\, G_F\, \frac{\rho}{M_p} Y_e \\ &= 3.8 \times 10^{-14} \, \left(\frac{\rho}{\text{g cm}^{-3}}\right) \, \left(\frac{Y_e}{0.5}\right) \text{ eV} \end{aligned}$$

(where Y_e is the electron number per nucleon).

FLAVOR EVOLUTION EQUATION

Schrödinger equation that controls the flavor evolution:

$$i\frac{\mathrm{d}}{\mathrm{d}x}\nu_\alpha = \mathcal{H}\,\nu_\alpha$$

In the presence of matter the flavor evolution of neutrinos is described by an effective Hamiltonian that is the sum of a free Hamiltonian plus a matter induced term that includes the effective potential and that therefore changes sign for neutrinos and antineutrinos:

$$\begin{aligned}\mathcal{H}(\nu) &= \mathcal{H}_0 + \mathcal{H}_{\mathrm{m}}\\ \mathcal{H}(\overline{\nu}) &= \mathcal{H}_0^* - \mathcal{H}_{\mathrm{m}}\end{aligned}$$

$$\mathcal{H}(\nu) = U \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} U^\dagger + \begin{bmatrix} V_e & 0 & 0 \\ 0 & V_\mu & 0 \\ 0 & 0 & V_\tau \end{bmatrix}$$

The subtraction from the Hamiltonian of a term proportional to the unit matrix, clearly does not change the flavor evolution, therefore all that counts for the matter term is a matrix of for $\mathcal{H}_{\rm n} = {\rm diag}[V, 0, 0]$.

It is clear that when:

$$V \gtrsim |E_j - E_k| = \frac{|\Delta m^2_{jk}|}{2E_\nu}$$

the ν flavor evolution is strongly modified.

It can be useful to note that the potential can be seen as having giving a different effective mass to the neutrinos.

$$E_\nu - V = \sqrt{p_\nu^2 + m^2} \simeq p_\nu + \frac{m^2}{2p_\nu}$$

and approximating $p_\nu \simeq E_\nu$::

$$m^2 \to m^2 + 2\, E_\nu\, V$$

with a shift in squared mass due to the matter

$$\begin{aligned}(\Delta m^2)^{\nu_e(\overline{\nu}_e)}_{\text{matter}} &= \pm 2\, E_\nu\, V \\ &\simeq 0.76 \times 10^{-4} \left(\frac{\rho}{\text{g cm}^{-3}}\right) \left(\frac{E_\nu}{\text{GeV}}\right) \frac{Y_e}{0.5}\ \text{eV}^2\end{aligned}$$

In general the matter density changes along the neutrino trajctory. If the density is constant however the flavor evolution can be solved analytically calculating the eigenvalues and eigenvectors of the (time independent) Hamiltonian.

Diagonalization of the Effective Hamiltonian in Matter

Study the case of two neutrino flavors.
Mixing ν_e–ν_μ or ν_e–ν_τ
The case is also relevant for $\theta_{13} = 0$
In this case the mixing is between ν_e and a linear combination of ν_μ and ν_τ.

Rediscuss for pedagogical reasons the vacuum case:
(with the compact notation $c = \cos\theta$, $s = \sin\theta$):

$$
\begin{aligned}
\mathcal{H}_0 &= U \ \mathrm{diag}[E_1, E_2] \ U^\dagger \\
&= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\
&= \frac{1}{2E} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} + \lambda \, \mathbf{I} \\
&= \frac{\Delta m^2}{4E} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} + \lambda' \, \mathbf{I} \\
&= \frac{\Delta m^2}{4E} \begin{bmatrix} -c^2 + s^2 & 2cs \\ 2cs & c^2 - s^2 \end{bmatrix} + \lambda' \, \mathbf{I} \\
&= \frac{\Delta m^2}{4E} \begin{bmatrix} -\cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & \cos 2\theta_0 \end{bmatrix} + \lambda' \, \mathbf{I}
\end{aligned}
$$

(the terms $\lambda \, \mathbf{I}$ or $\lambda' \, \mathbf{I}$ are proportional to $\mathbf{I}$ and can be dropped.

General Hamiltonian 2×2

The matter Hamitonian, that we can recast in the form:

$$\mathcal{H}_{\rm mat} = \begin{bmatrix} V_e & 0 \\ 0 & V_\mu \end{bmatrix}$$

$$= \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} + \lambda'' \, \mathbf{I}$$

$$= \tfrac{V}{2} \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} + \lambda''' \, \mathbf{I}$$

terms like $\lambda \, I$ can be dropped

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\rm mat} \\ &= \frac{\Delta m^2}{4E_\nu} \begin{bmatrix} -\cos 2\theta_0 + a & \sin 2\theta_0 \\ \sin 2\theta_0 & +\cos 2\theta_0 - a \end{bmatrix} \end{aligned}$$

where:

$$a = \frac{2VE}{\Delta m^2} = \frac{2\sqrt{2} G_F \, N_e}{\Delta m^2}$$

It is now straightforward to diagonalise the matter hamiltonian, comparing with the form of the free one.

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\rm mat} \\ &= \frac{\Delta m^2}{4E_\nu} \begin{bmatrix} -\cos 2\theta_0 + a & \sin 2\theta_0 \\ \sin 2\theta_0 & +\cos 2\theta_0 - a \end{bmatrix}\end{aligned}$$

$$= \quad \frac{(\Delta m^2)_{\rm eff}}{4E_\nu} \begin{bmatrix} -\cos 2\theta_{\rm m} & \sin 2\theta_{\rm m} \\ \sin 2\theta_{\rm m} & +\cos 2\theta_{\rm m} \end{bmatrix}$$

where

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta_0}{\sin^2 2\theta_0 + (\cos 2\theta_0 - a)^2}$$

$$(\Delta m^2)_{\rm eff} = \Delta m^2 \times \sqrt{\sin^2 2\theta_0 + (\cos 2\theta_0 - a)^2}$$

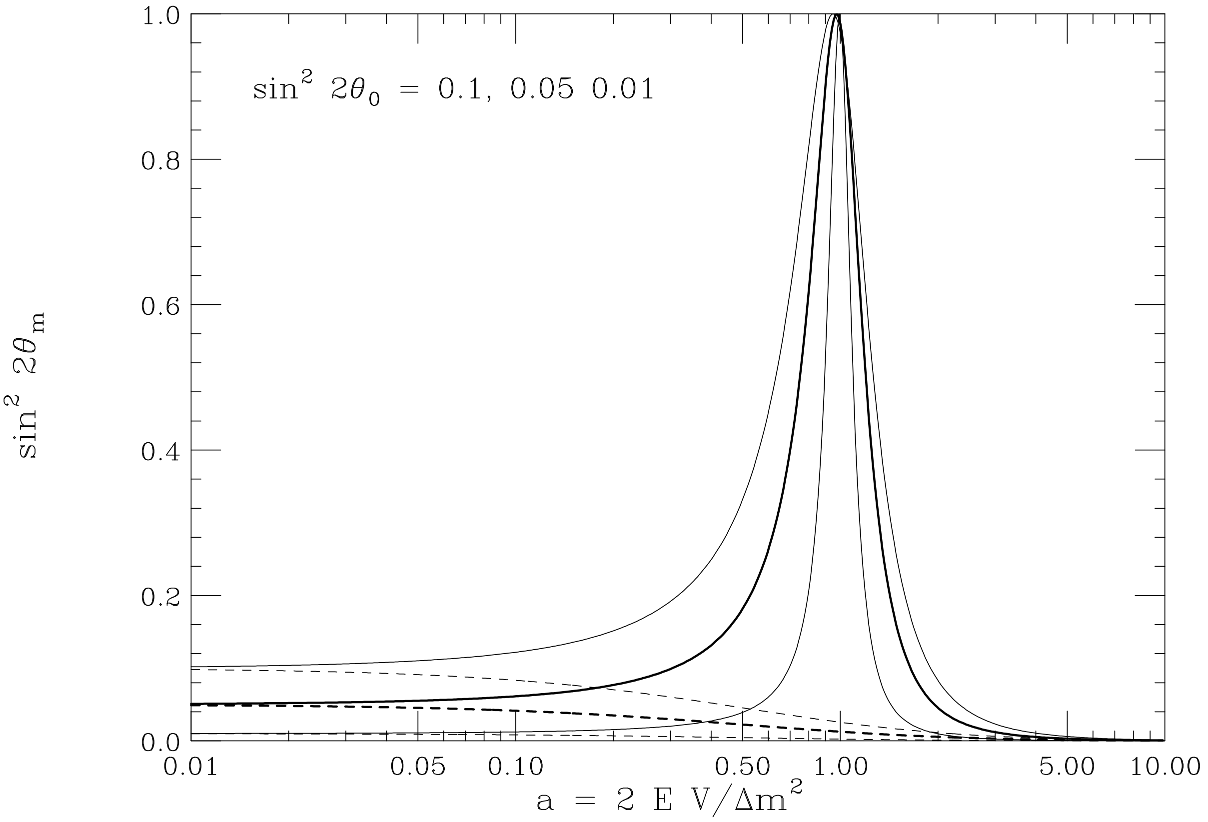
The effective mixings in matter of neutrinos and anti–neutrinos are different because ν and $\overline{\nu}$ have opposite effective potentials.
The eigenvalues/eigenvectors solutions for anti–neutrinos can be obtained with the replacement $V \to -V$, that is $a \to -a$.

Note that without loss of generality we can assume that Δm^2 is positive[1] letting θ vary in the interval $\theta \in [0, \pi/2]$; then $a > 0$ corresponds to ν mixing, and $a < 0$ corresponds to $\overline{\nu}$ mixing.

[1] For two flavor mixing is also possible to use a different convention, of reducing the range of variability of θ in the range $[0, \pi/4]$, coinsidering both positive and negative Δm^2. In this convention the ν_1 is always the state with the largest overlap with ν_e; in the convention used in the text ν_1 is the eigenstate with the lowest mass.

The MSW resonance

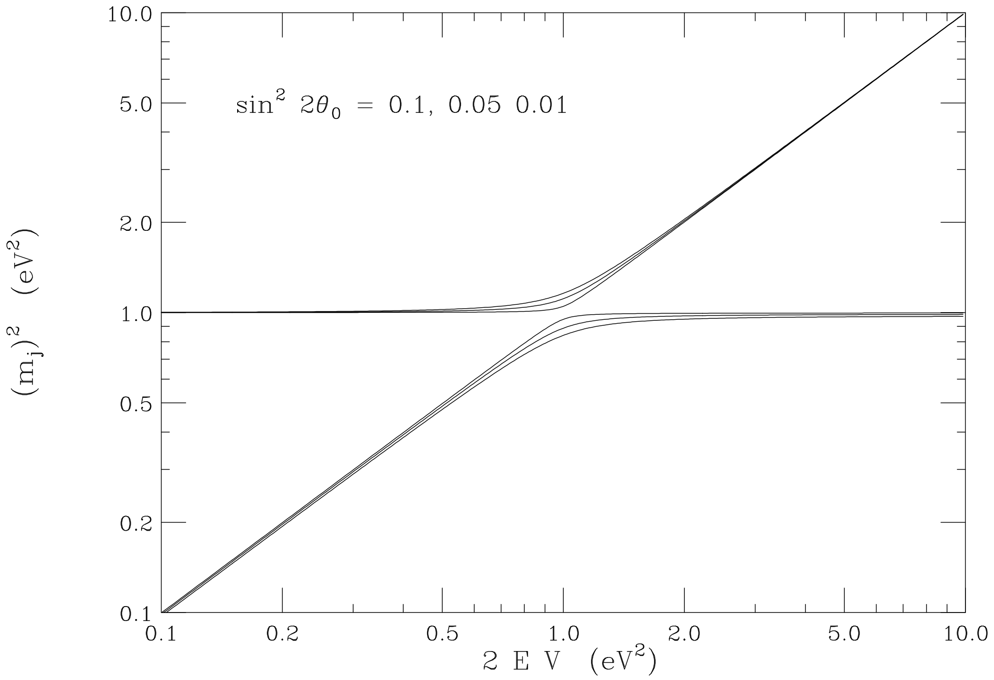
It is very instructive to study the behaviour of effective mixing parameters in matter as a function of the quantity $a = 2VE/\Delta m^2$. Note that a is proportional to the density V and to the neutrino energy E.



MSW : Initials of

Mikheyev and Smirnov, [Sov. Jour. Nucl. Phys. **42**, 913 (1985)]

Wolfenstein, [Phys. Rev. D**17**, 2369 (1978).]



Effective squared masses in matter for neutrinos in the case of two flavor ν_e–ν_μ (or ν_e–ν_τ) mixing. The mass eigenvalues are taken as $m_1^2 = 0.1$ eV^2 and $m_2^2 = 1$ eV^2. The different lines correspond to different values of $\sin^2 2\theta$ (with $\theta < 45°$). The smallest (largest) value of $\sin^2 2\theta$ correspond to the smallest (largest) $\Delta m^2_{\rm eff}$.

Limit $\theta \to 0$

$$
\begin{aligned}
m_1^2 &= m_1^2 \\
m_2^2 &= m_2^2 + 2VE_\nu
\end{aligned}
$$

Qualitive discussion of the effective mixing

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta_0}{\sin^2 2\theta_0 + (\cos 2\theta_0 - a)^2}$$

1. The low density limit can be obtained for $a \to 0$, it is easy to see thaT one recovers the vacuum case.

2. The limit $|a| \to \infty$ corresponds to very high energy or very large density. The limiting behaviour corresponds to

$$\sin^2 2\theta_{\mathrm{m}}^{\nu(\overline{\nu})} \to \frac{1}{a^2} \simeq \left(\frac{\Delta m^2}{2EV}\right)^2 \to 0$$

$$(\Delta m^2)_{\mathrm{eff}}^{\nu(\overline{\nu})} \to \Delta m^2 \times a = \pm 2EV$$

It is possible to see that effective mixing in matter corresponds to $\theta_m \to \pi/2$ for neutrinos and to $\theta_m \to 0$ for anti–neutrinos. It is simple to see that the ν_e or $\overline{\nu}_e$ in very high density matter are an eigenvector of propagation, because of its very large potential; the ν_e (with $V > 0$) corresponds to the heaviest eigenvector $|\nu_{2,\mathrm{m}}\rangle$, while $\overline{\nu}_e$ (with $V < 0$) corresponds to the lightest eigenvector $|\nu_{1,\mathrm{m}}\rangle$.

3. **The "RESONANCE"**
   The mixing parameter $\sin^2 2\theta_\mathrm{m}$ becomes unity (that is θ_m becomes $\pi/4$) for the condition

   $$a = \cos 2\theta_0$$

   that is:

   $$2V\,E = \Delta m^2 \cos 2\theta_0$$

   This condition can be satisfied with $a > 0$ (that is for neutrinos) when $\theta < \pi/4$; and with $a < 0$ (that is for anti–neutrinos) when $\theta > \pi/4$.

This situation can be pictured geometrically for example observing that the eigenvectors that the mixing angle θ_m^ν for neutrinos "rotates" monotonously when the density increases from the vacuum value θ (for a vanishingly small density) toward increasing values reaching asymptotically the value $\theta_{\rm m}^{\nu} \to \pi/2$ for very large density; for antineutrinos the mixing angle $\theta_{\rm m}^{\overline{\nu}}$ rotates in the opposite direction toward the asymptotic value $\theta_{\rm m}^{\overline{\nu}} \to 0$. Clearly the mixing angle in matter must become "maximal" (that is take the value $\theta_{\rm m} = 45^\circ$) for neutrinos *or* antineutrinos, for a particular value of the product $E_\nu\, N_e$. This condition will be present for neutrinos when $\theta < 45^\circ$ that is when the lighest mass state is predominantly a ν_e as it is naively expected "normal" ordering of the neutrino masses, of for anti–neutrinos if $\theta > 45^\circ$ that is for an inverted hierarchy of the ν masses.

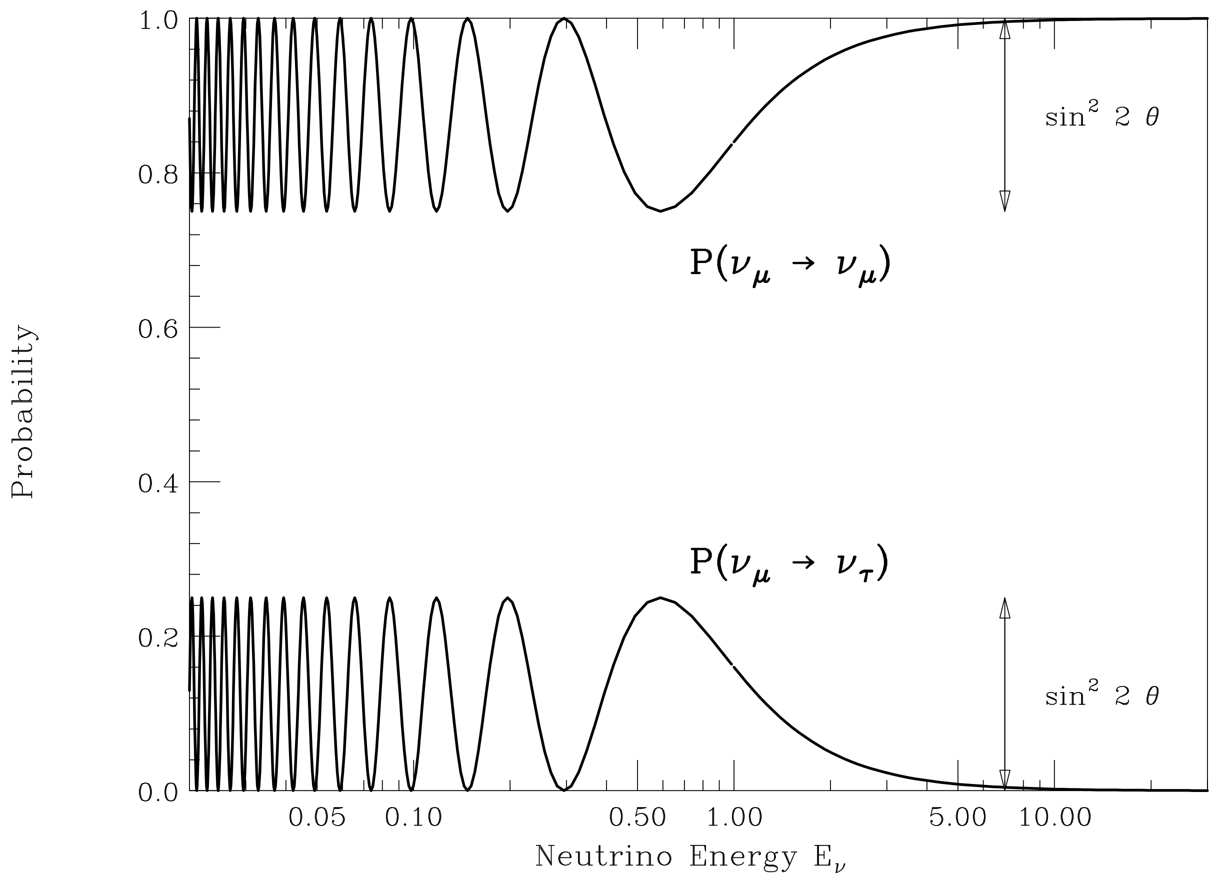


Figure 1: Example of the shape of the the oscillation probabilities $P(\nu_\mu \to \nu_m u)$ and $P(\nu_\mu \to \nu_\tau)$ as function of the neutrino energy E_ν for a fixed neutrino pathlength L. In the example we have chosen $\Delta m^2 = 3 \times 10^{-3}$ eV^2 (approximately the value indicated by the atmospheric neutrino experiments) and $L = 730$ Km (the ν pathlength of the Fermilab to Minos and Cern to Gran Sasso projects).

Illustration of $\nu_\mu \leftrightarrow \nu_e$ oscillations in homogeneous matter.