Matter effects
for Solar Neutrinos

Flavor Evolution equation

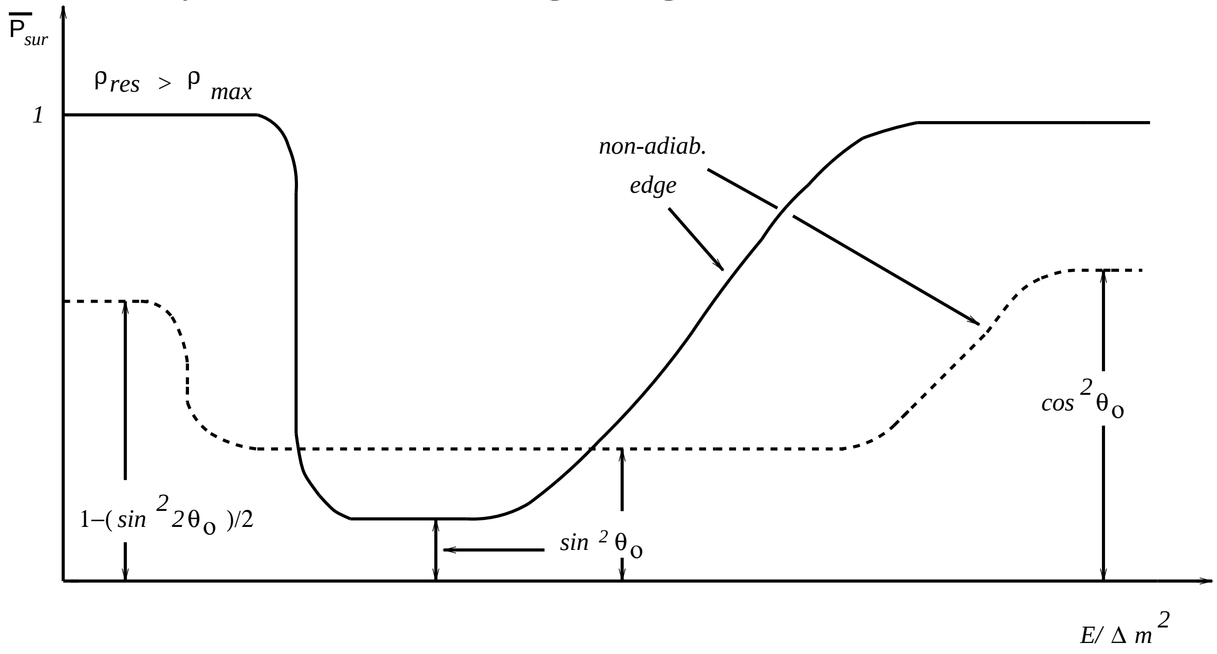
$$
\begin{aligned}
i\frac{d}{dx}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} &= [\mathcal{H}_0 + \mathcal{H}_{\mathrm{mat}}(x)]\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} \\
&= \left\{\frac{1}{2E}\,U\begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2\end{bmatrix}U^\dagger \right. \\
&\quad \left. + \begin{bmatrix} V(x) & 0 \\ 0 & 0\end{bmatrix}\right\}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}
\end{aligned}
$$

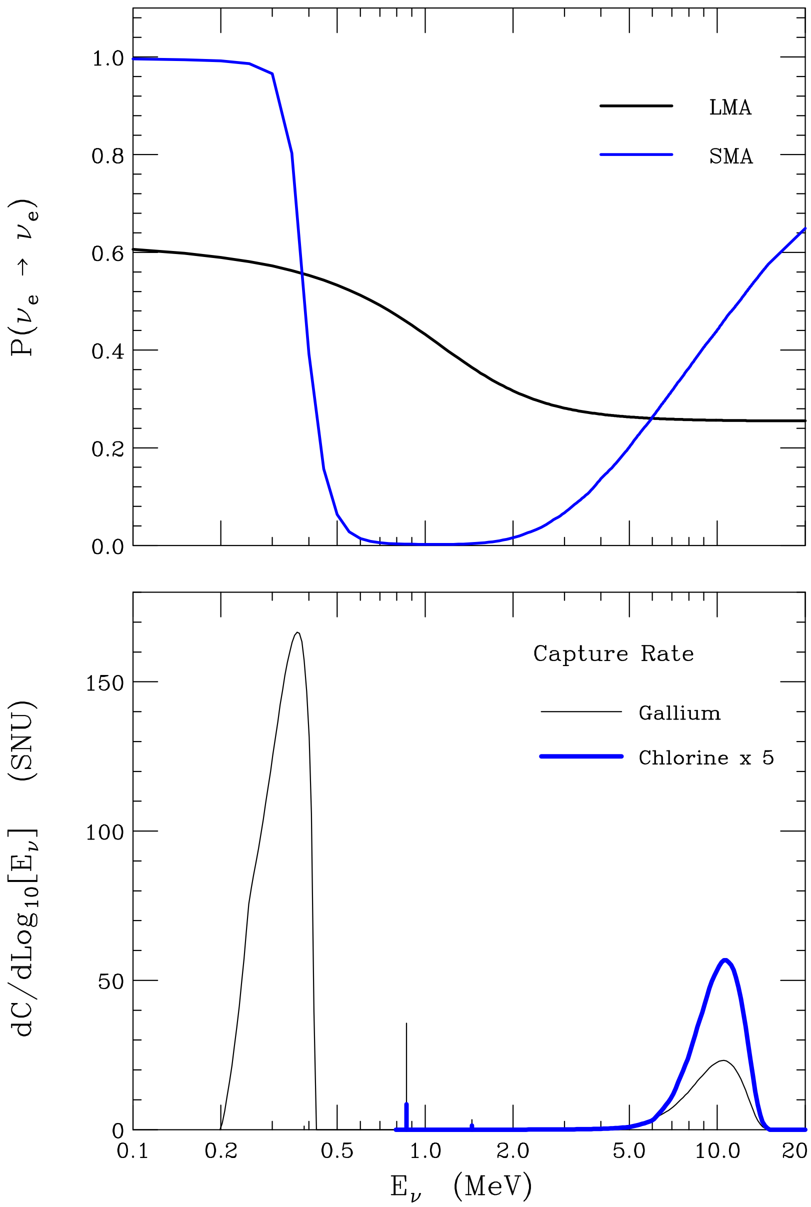
where $V(x) = V_e(x) - V_\mu(x) = \sqrt{2}G_F\,N_e(x)$ is the effective matter potential.

The ν_e survival probability can be calculated integrating this differential equation along a straight line trajectory from an initial point close to the sun center ($r = 0$) taking into account the known density of the sun. The initial condition is a pure ν_e state, and the survival probability is obtained squaring the amplitude of the ν_e component after the neutrino has emerged to the surface.

The "bathtub" shape of the oscillation probability

The $P_{\nu_e\to\nu_e}$ probability takes a remarkable form (see fig. **??** and fig. 6), that exhibit a large suppression in a well defined energy range, even for very small mixing angles.





Top panel: Energy dependence of the oscillation probabilities that fit the results of the solar neutrino experiments for the Small Mixing Angle Solution (SMA), and the Large Mixing Angle (LMA) solution.

The bottom panel show the contribution to the solar neutrino capture rate for Gallium–71 and Chlorine–37.

It is possible to understand qualitatively the shape of the oscillation probability when matter effects are important with some simple considerations. One can distinguish three regions for the oscillation probability:

$$P_{\nu_e \to \nu_e}(E) \simeq \begin{cases} 1 - \frac{\sin^2 2\theta}{2} & \text{for } E < E_A \\ \\ \sin^2\theta & \text{for } E_A < E < E_B \\ \\ \to \cos^2\theta & \text{for } E \gg E_B \end{cases}$$

- **Small matter effects**
- **Adiabatic region**
- **Non–Adiabatic region**

Small matter effects
The first (low energy) region simply corresponds to the vacuum oscillation probability (after average over fast oscillations).

Adiabatic region
After a sharp fall there is the remarkable fact that in a certain energy interval the oscillation probability is flat (no energy dependence) with the value $P_{\nu_e \to \nu_e} = \sin^2 2\theta$. This seems very surprising, because it says that the smaller is the mixing, the smaller is the survival probability, that is the larger it is the probability to have a flavor transition. This flat region is also known as the "adiabatic region". The third region where the survival probability grows slowly is known as the "non–adiabatic" region The existence of the interval of energy where $P_{\nu_e \to \nu_e}$ is small can be understood qualitatively very easily with the following observations:

1. Each solar neutrino is created near the center of the sun, where the density is very

large. Using the equation:

$$E_\nu - V = \sqrt{p^2 + m^2} \simeq p + \frac{m^2}{2p}$$

we can observe that the potential acts as a positive contribution to the effective squared mass of ν_e in matter $+2E_\nu V$ (we have used the approximation $p \simeq E_\nu$). Clearly then if $2E_\nu V$, is suficiently large ($2E_\nu V_{\text{center}} \gg \Delta m^2$) the electron neutrino is approximately a matter eigenstate with a large effective squared mass: $m^2_{\text{eff}} \simeq 2E_\nu V_{\text{center}}$, that is the heaviest mass eigenstate. In other words $|\nu_e\rangle \simeq |\nu_{2,m}\rangle$.

2. When the neutrino travels outward toward the surface of the star the eigenstates are modified according to the change of the (decreasing) density. If the change is sufficiently slow the neutrino state evolves *remaining the heavier effective mass eigenstate*. This is analogous of the quantum

mechanical problem of an Hamiltonian that depends on one parameter x (for example the value of the magnetic field) that can be varied continuosly. The eigenvalues of this Hamiltonian can be calculated as a function of this parameter as $E_n(x)$ where n labels the different states ($n = 0$ is the ground state, $n = 1$ the first excited state and so on). If set the parameter at an initial value x_i, prepare the system in the eigenstate $|m(x_i)\rangle$ (with energy $E_m(x_i)$) and then *slowly* or "adiabatically" change the parameter from x_i to x_f, it can be demonstrated, that the system at the end of the process will be in the quantum state $|m(x_f)\rangle$ with energy $E_m(x_f)$, that is it will remain on the same energy level, without "jumping" to a different level.

3. A ν_e created near the center of the sun will then emerges as a state $|\nu_{2,\text{vacuum}}\rangle \equiv |\nu_2\rangle$.

Since $|\nu_2\rangle = \sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle$ it follows that solar neutrinos (created as ν_e) emerge at the surface of the sun with a probability $\sin^2\theta$ of still being of the same flavor.

The sharp edge at $E \simeq E_A$ can be understood as the value such that

$$2E_A V_{\rm center} \simeq \Delta m^2.$$

For $E_\nu \lesssim E_A$ ν_e created at the center of the Earth are not approximately equal to $|\nu_{2,m}\rangle$ and the argument does not apply. For completeness we can observe that it is simple to describe more precisely the oscillation probability in the adiabatic regime noting that if θ_m is the mixing angle near the center of the sun, then a ν_e created at the center of the sun will be in a state:

$$|\nu_e\rangle = \cos\theta_m|\nu_{1,m}\rangle + \sin\theta_m|\nu_{2,m}\rangle$$

$$|\nu_f\rangle = \cos\theta_m\,|\nu_1\rangle + \sin\theta_m\,|\nu_2\rangle$$

The survival probability $P_{\nu_e \to \nu_e}$ is then

$$\begin{aligned} P_{\nu_e \to \nu_e} &= |\langle \nu_e | \nu_f \rangle|^2 \\ &= (\cos\theta \cos\theta_m + \sin\theta \sin\theta_m)^2 \end{aligned}$$

For $\theta_m \to \pi/2$, that corresponds to $|\nu_{2,m}\rangle \to |\nu_e\rangle$, the probability goes to limit discussed before of $\sin^2 2\theta$.

Non–Adiabatic

To understand the non–adiabatic part of the oscillation probability we can start with the observation that clearly the arguments outlined above are not valid is the density changes too rapidly.

As a simple example let us consider the case where the density changes abruptly from a very high value to zero. In this case the a small mixing angle would oviously correspond $P_{\nu_e \to \nu_e} = 1$ and not $\sin^2 2\theta$ since the mixing would vanish both in matter ($\theta_m \simeq \pi/2$) and in vacuum ($\sin\theta \simeq 0$). This corresponds to the situation of the example we discussed before of the Hamiltonian of a quantum mechanical system that depends on one parameter when the parameter is changed rapidly, in this case a state can "jump" discontinuosly from one quantum level to a different one. To understand qualitatively when the density variation is slow or rapid let us go back to the formula of effective mixing in matter:

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{(\cos 2\theta - A(x))^2 + \sin^2 2\theta}$$

with $A(x) = 2EV(x)/\Delta m^2$. The flavor transition can only take place when the mixing is large, that is when $\sin^2 2\theta_m$ is close to unity or "at the resonance". The potential that corresponds to $\sin^2 2\theta_m = 1$ is given by $A = \cos 2\theta$ or

$$V(r) = V^* = \frac{\Delta m^2}{2E} \cos 2\theta.$$

and the range of potential that correspond to $\sin^2 2\theta_m > 0.5$

$$V(r) = V^* [1 \pm \sin 2\theta]$$

The density of the sun, and therefore the potential depends exponentially on the radius as $V(r) \simeq V_{\rm center} e^{-r/R_0}$ with $R_0 \simeq 0.1\ R_\odot$. Therefore the resonance potential corresponds to radius $r^* = R_0 \log[V_c/V^*]$, and the region where the mixing is larger than 0.5 to the interval $V \in [V_-, V_+]$ with

$$V_\pm = V^* \left[1 \pm \sin^2 \theta\right]$$

that corresponds to a radius interval

$$\begin{aligned}(\Delta r)_{\rm res} &= R_0[\log(1 + \sin 2\theta) - \log(1 - \sin 2\theta)] \\ &\simeq 2 R_0 \sin^2 \theta\end{aligned}$$

This interval mus be longer that the oscillation length at the resonance:

$$\lambda_{\rm res} = \frac{4\pi E}{\Delta m^2 \sin 2\theta}$$

The condition $(\Delta r)_{\rm res} \gtrsim \lambda_{\rm res}$ results

$$E \lesssim E_B \simeq \lambda_{\rm res} = \frac{R_0}{2\pi} \Delta m^2 \sin^2 2\theta$$

Note that the energy that separates the adiabatic regime from the non adiabatic regime is proportional to the product: $E_B \propto \Delta m^2 \sin^2 2\theta$.

Triangles in the $(\sin^2 2\theta, \Delta m^2)$ plane

It is not difficult to see that a fixed average suppression for the detection of solar neutrinos Data/Prediction $= f$, can then be explained by a "triangle like" curve in the $(\sin^2 2\theta, \Delta m^2)$ plane (see fig. 4) The triangle has three sides, the "horizontal side" corresponds to a constant value for Δm^2, that is to a fixed value of the energy $E_A \propto \Delta m^2$ that marks the lower limit of the energy range for adiabatic transitions; the "diagonal side" corresponds to a fixed value of the energy $E_B \propto \Delta m^2 \sin^2 2\theta$ that marks the upper limit of the same range; and finaly the "vertical band" corresponds to a constant value of the mixing angle, and therefore to an approximately constant value of the survival probability (in the adiabatic regime).

Combined results

The analysis of the total rate of each solar neutrino experiment results in an allowed region in the plane $(\sin^2 2\theta, \Delta m^2)$ of triangular form in the region where matter effect are present, plus a region at much lower Δm^2 that corresponds to the vacuum solution. A combined analysis of the total rates observed by the radiochemical experiments (Chlorine and Gallium) and by the electron scattering experiments with Water Cherenkov detectors, let survive several disjoint "patches" of allowed values in the parameter space that are shown in fig. 5. The three patches in the regions where matter effects are important are known as the "Large Mixing Angle" (LMA), the "Small Mixing Angle" (SMA), and the "Low" solution. The vacuum solution has a complex shape that is subdivided in several sub–patches.

The shape of the oscillation probability in the regions of the LMA and SMA solutions is shown in the top panel of figure 6, and compared with the sensitivity of the Chlorine and Gallium experiments. One can see how these probabilities can qualitatively reproduce the experimental results, that indicate a larger suppression for the Chlorine experiment.

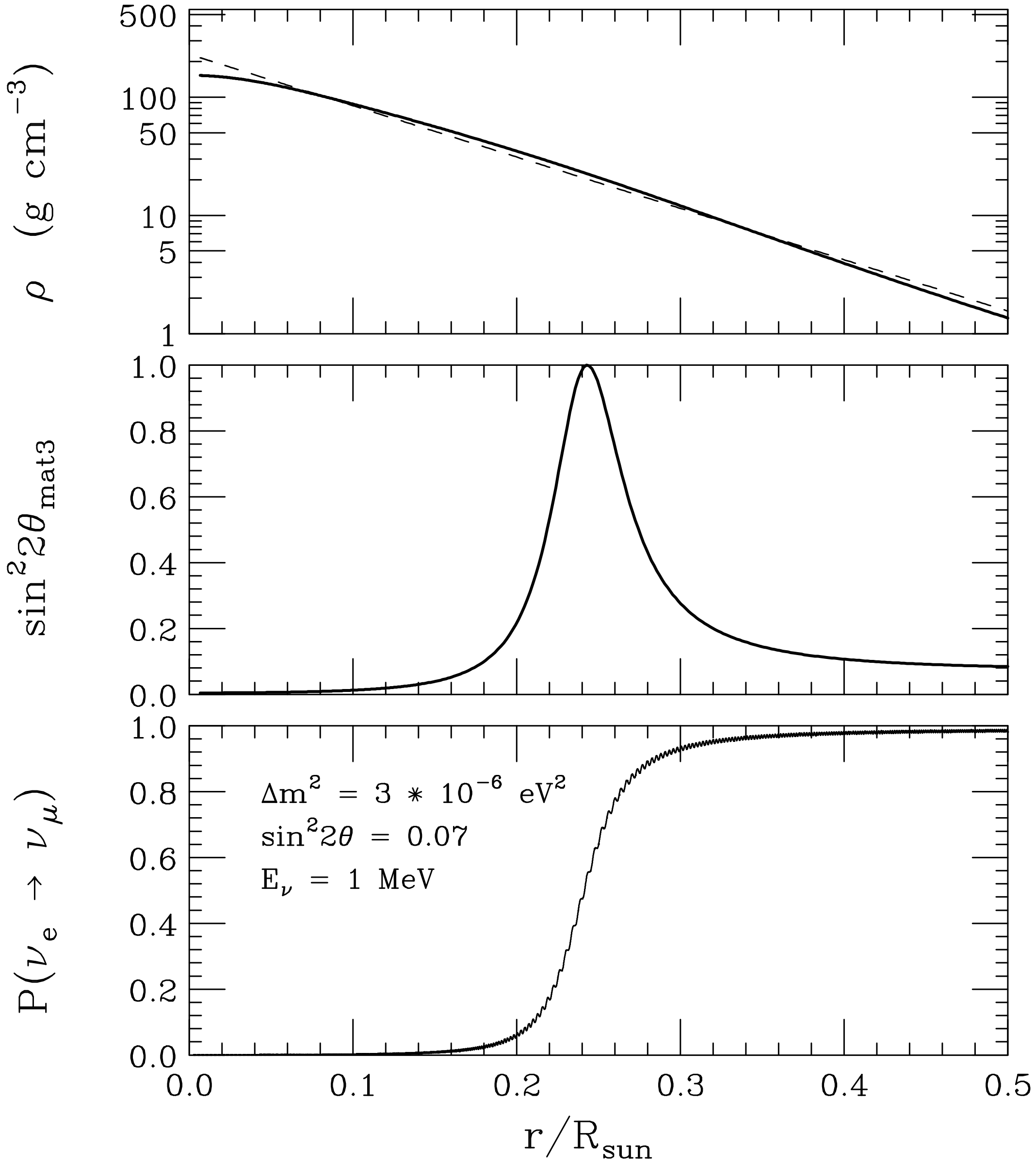


Figure 1: This figure illustrates the flavor evolution of a ν_e created at the center of the sun. For illustration we have chosen the values of $\Delta m^2 = 3 \times 10^{-6}$ eV^2, $\sin^2 2\theta = 0.07$ and $E_\nu = 1$ MeV The top panel shows the sun density as a function of the radius (solid line); the density can be approximated as an exponential (dashed line). The middle panel shows the effective mixing parameter $\sin^2 2\theta_{\rm m}$ as a function of r. Mixing is large in a layer of the sun. The bottom panel shows the square of the amplitude $|A(\nu_e \to \nu_\mu)|^2$ as a function of radius. The magic of the MSW mechanism is in action.

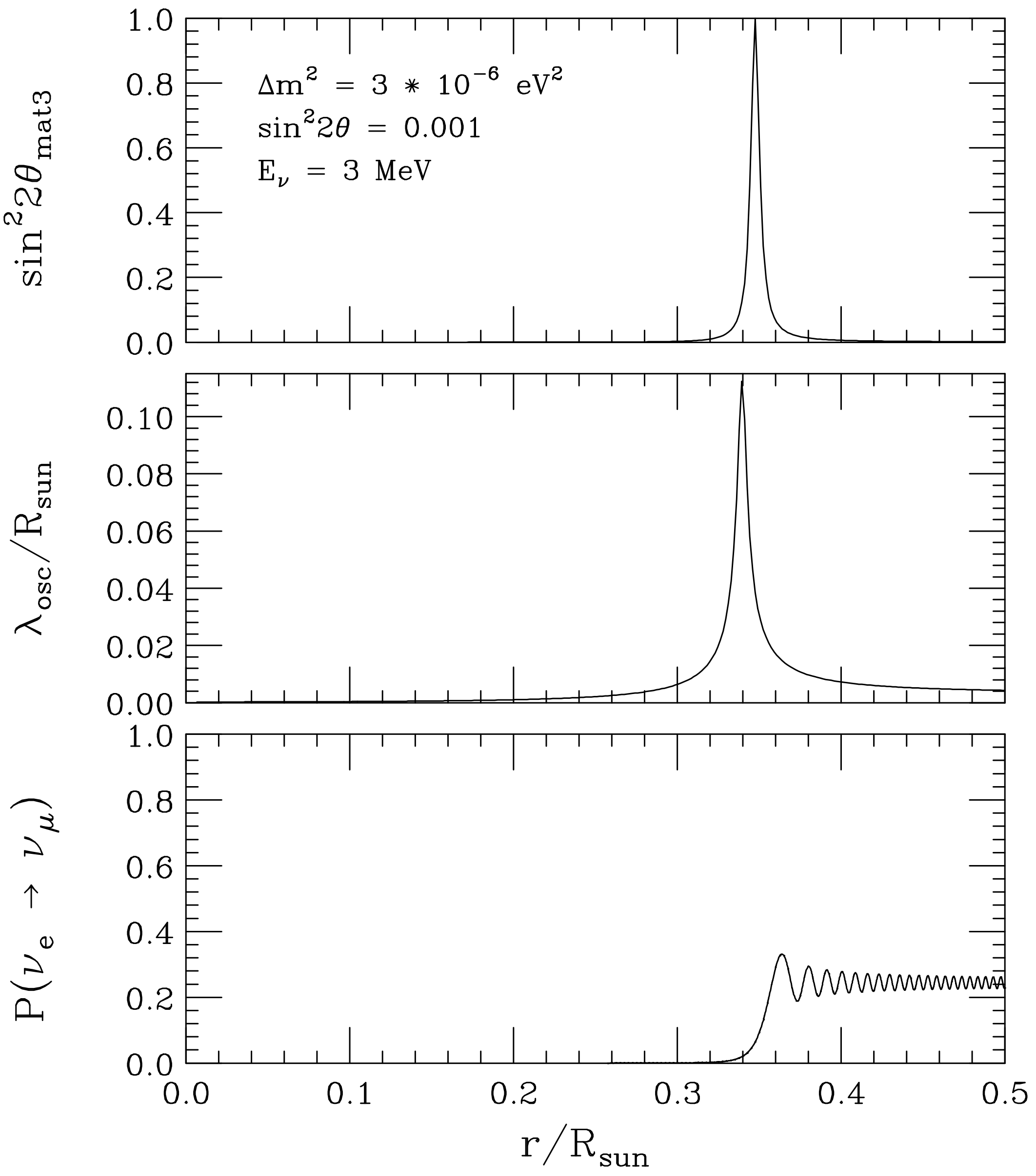


Figure 2: Flavor evolution of a ν_e created at the center of the sun. For illustration we have chosen the values of $\Delta m^2 = 3 \times 10^{-6}$ eV^2, $\sin^2 2\theta = 10^{-3}$ and $E_\nu = 3$ MeV. The top panel shows the effective mixing parameter $\sin^2 2\theta_{\rm m}$ as a function of r; the middle panel shows the oscillation length in matter; the bottom panel shows the square of the amplitude $|A(\nu_e \to \nu_\mu)|^2$. The layer of the sun where the mixing is close to maximal is too narrow for complete transition to ν_μ. This corresponds to the "non–adiabatic regime.

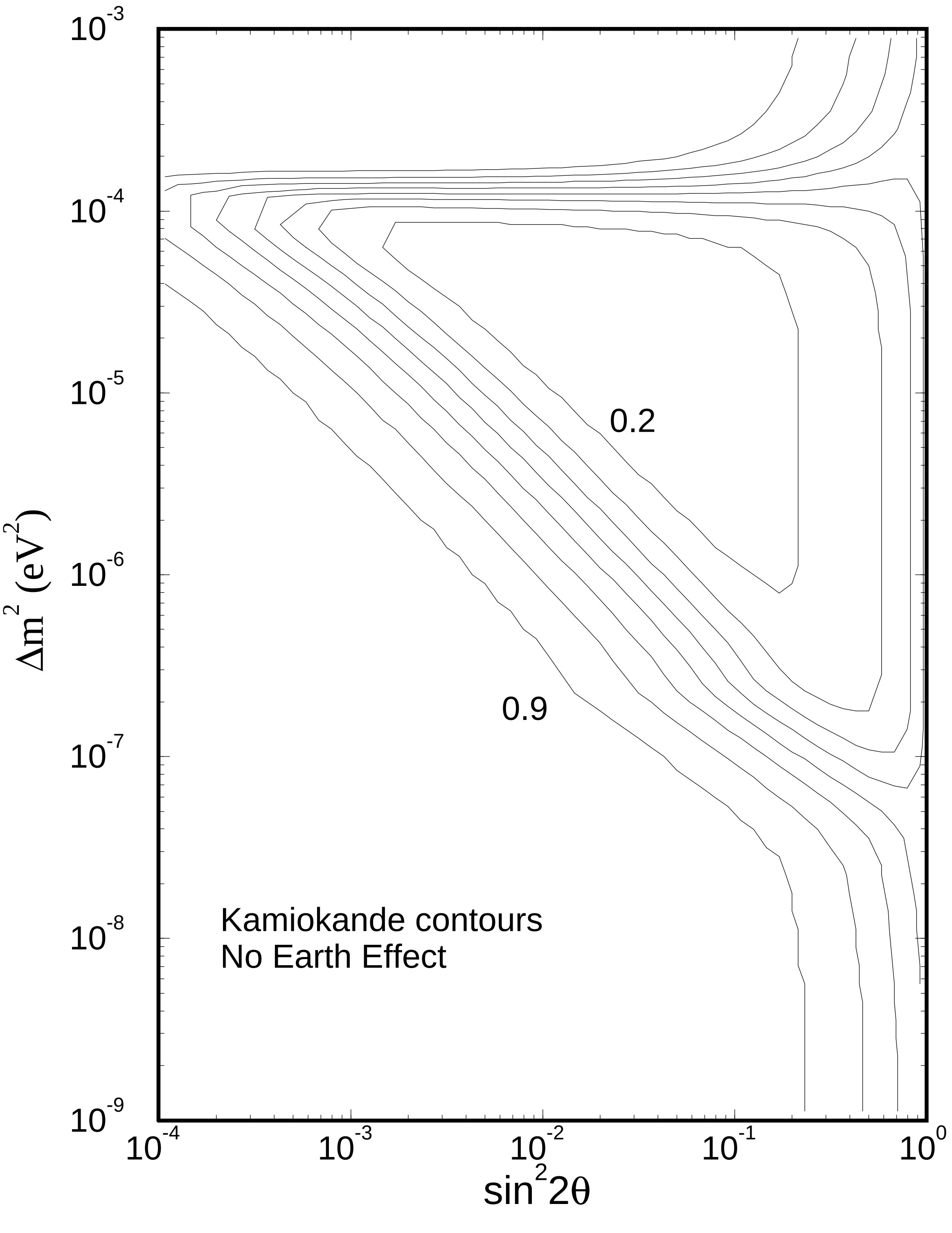


Figure 3: Lines in the plane Δm^2–$\sin^2 2\theta$ for ν_e–ν_μ (or ν_e–ν_τ) mixing, that correspond to a constant suppression of the rate in a water Cherenkov detector.