Accelerator Physics and Neutrino Beams

I. Linear Theory of Perfect Machines

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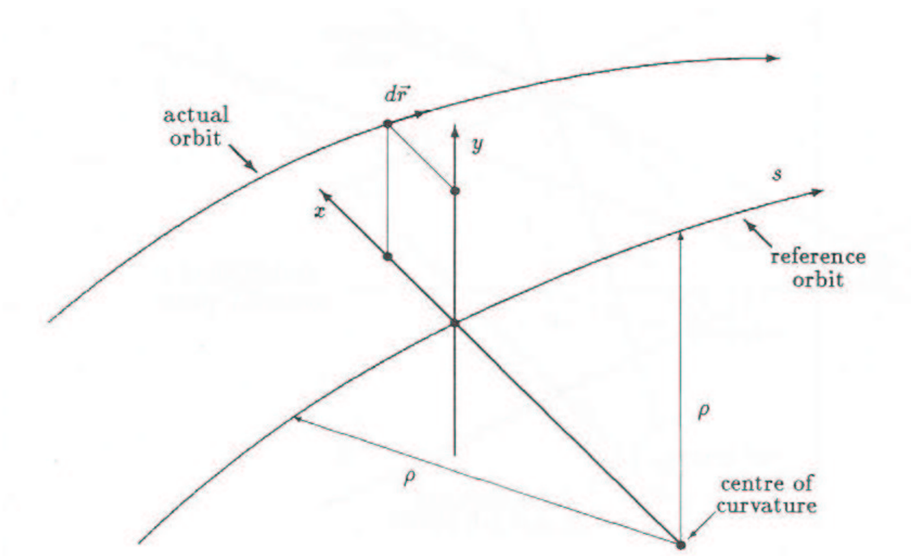
`http://keil.home.cern.ch/keil/MuMu/Doc/NuSchool05/talk1.pdf`

Programme of First Lecture – Linear Theory of Perfect Machines

- Coordinate system and equilibrium orbit
- Linear equations of motion
- Stability of betatron oscillations
- Amplitude of betatron oscillations
- Phase space, admittance, emittance
- Dispersion

Coordinate System and Equilibrium Orbit



- Curvilinear system: Arcs of circle with radius ρ in dipoles, straight lines in all other elements
- Distance s along reference orbit
- Horizontal displacement x, vertical displacement y
- Displacement u may be either horizontal or vertical
- Reference orbit in median plane in all examples
- Reference orbit closes on itself

Linear Equations of Motion

- Get force from Lorentz equation, neglect all terms of order > 1, and find equations of motion in horizontal coordinate x and vertical coordinate y

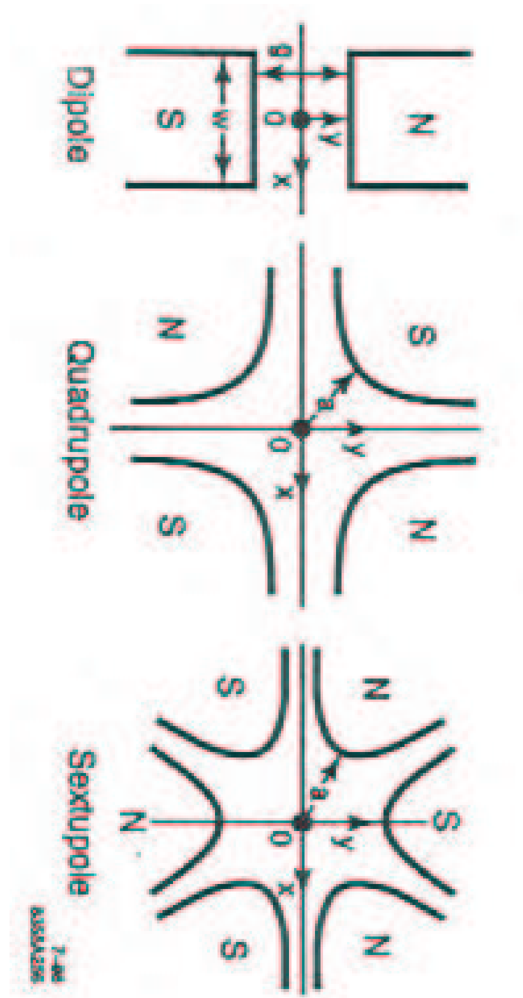
$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left[\rho^{-2}(s) - K(s)\right] x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + K(s) y = 0$$

with radius of curvature $\rho(s)$, momentum error of particle $\frac{\Delta p}{p}$, and focusing strength

$$K(s) = -\frac{1}{B\rho} \frac{\partial B_y}{\partial x}$$

- Magnetic rigidity $B\rho = p/e$ is a constant of the motion in purely magnetic field
- Linear coupling terms which link the two equations omitted



Stability of Beatron Motion I

- Particles launched with small offsets x and y and small angles x' and y' with respect to reference orbit execute betatron oscillations
- Use $\frac{\Delta p}{p} = 0$, and find that both equations of motion have the same form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + K(s)u = 0$$

  where the meaning of $K(s)$ depends on the plane under consideration, and the coordinate u may be either x or y
- The focusing strength $K(s)$ is periodic with the circumference C
- Solution of any second-order differential equation can be written as

$$\begin{aligned} u(s) &= C(s, S_0)u(s_0) + S(s, s_0)u'(s_0) \\ u'(s) &= C'(s, S_0)u(s_0) + S'(s, s_0)u'(s_0) \end{aligned}$$

- Cos-like function C and sin-like function S depend on s_0 and s, normalised such that $C(s_0, s_0) = S'(s_0, s_0) = 1$ and $C'(s_0, s_0) = S(s_0, s_0) = 0$, where prime $'$ denotes $\mathrm{d}/\mathrm{d}s$

Stability of Beatron Motion II

- Write equations for $u(s)$ and $u'(s)$ with matrix $M(s|s_0)$

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = M(s|s_0) \begin{pmatrix} u(s_0) \\ u'(s_0) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} u(s_0) \\ u'(s_0) \end{pmatrix}$$

- Properties of machine lattice in M and initial conditions nicely separated
- Determinant of M is Wronskian W of C and S and a constant of the motion, its value is $W = 1$ by normalisation
- Successive application for positions $s_1, s_2, \ldots, s_i$ shows that $u(s_i)$ and $u'(s_i)$ are related to $u(s_0)$ and $u'(s_0)$ by a product of matrices M
- For $K(s)$ constant for $s_0 \leq s \leq s_1$, matrix M becomes explicitly

$$M(s_1|s_0) = \begin{pmatrix} \cos\varphi & K^{-1/2}\sin\varphi \\ -K^{1/2}\sin\varphi & \cos\varphi \end{pmatrix}$$

with $\varphi = K^{1/2}(s_1 - s_0)$

Stability of Betatron Motion III

- For $K(s) < 0$ a more convenient form of M with $\varphi = (-K)^{1/2}(s_1 - s_0)$ is

$$M(s_1|s_0) = \begin{pmatrix} \cosh\varphi & (-K)^{-1/2}\sinh\varphi \\ (-K)^{1/2}\sinh\varphi & \cosh\varphi \end{pmatrix}$$

- Sufficient condition for stability of betatron oscillations: C and S are bounded for all s
- Study matrix $M = M(s_0 + L|s_0)$ for a full period of length L
- Any (2×2) matrix with unity determinant can be written as follows

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

where α, β and $\gamma = (1+\alpha^2)/\beta$ are formal parameters for the time being, whose physical significance will become apparent later on

Stability of Betatron Motion IV

- By induction, it can be shown that M^k becomes

$$M^k = \begin{pmatrix} \cos k\mu + \alpha \sin k\mu & \beta \sin k\mu \\ -\gamma \sin k\mu & \cos k\mu - \alpha \sin k\mu \end{pmatrix}$$

- Elements of M^k are bounded for all k, and betatron oscillations are stable, if μ is real, and the trace of M satisfies $|Tr(M)| \leq 2$
- Phase angle μ related to the eigenvalues of M by $\lambda = \exp(\pm i\mu)$
- For real μ, the λ are a complex conjugate pair on the unit circle
- For imaginary μ, the λ are a reciprocal pair on the real axis

Amplitude of Betatron Oscillations I

- Floquet's theorem states that the equation of motion with periodic K has always two particular solutions of the form

$$\begin{aligned} u_1(s) &= p_1(s)\exp(+i\mu s/L) \\ u_2(s) &= p_2(s)\exp(-i\mu s/L) \end{aligned}$$

  with functions p_1 and p_2 periodic with period L

- It follows that $u_i(s+L) = u_i(s)\exp(\pm i\mu)$
- On the other hand

$$u_i(s+L) = u_i(s)(\cos\mu + \alpha\sin\mu) + u_i'(s)\beta\sin\mu$$

- Hence u_i must satisfy the first-order differential equation

$$u_i\alpha + u_i'\beta = \pm i u_i$$

- Obtain by differentiation

$$\frac{u_i"}{u_i'} - \frac{u_i'}{u_i} = -\frac{\alpha'}{\pm i - \alpha} - \frac{\beta'}{\beta}$$

Amplitude of Betatron Oscillations II

- Second equation for the same quantities follows from equation of motion

$$\frac{u_i''}{u_i'} - \frac{u_i'}{u_i} = -\frac{K\beta}{\pm i - \alpha} - \frac{\pm i - \alpha}{\beta}$$

- Equating the r.h.s. of the two equations and ordering terms yields

$$(\alpha^2 + K\beta^2 + \alpha\beta' - \alpha'\beta - 1) \pm i(2\alpha + \beta') = 0$$

- When the stability criterion is satisfied, all terms inside brackets are real, and real and imaginary part must satisfy the relations

$$\beta' = -2\alpha \qquad \alpha' = K\beta - \gamma$$

- Using $\beta' = -2\alpha$ in the equation for u_i'/u_i yields

$$u_i'/u_i = (\pm i + \beta'/2)/\beta$$

- Integrating this equation yields the solution

$$u_i(s) = a\beta^{1/2}\exp(\pm i\mu(s))$$

where a is dtermined by the initial conditions and $\mu(s) = \int \mathrm{d}s/\beta(s)$

Amplitude of Betatron Oscillations III

- Betatron oscillations behave like quasi-harmonic oscillators with an instantaneous amplitude proportional to $\sqrt{\beta}$ and an instantaneous wavelength $\lambda = 2\pi\beta$
- $\mu(s)$ is generalisation of $\cos\mu = Tr(M)/2$ which defines μ only modulo 2π.
- Integrating it over the whole cicumference C yields the number of betatron oscillations in a turn, i.e. the tune Q

$$Q = \frac{1}{2\pi} \int_s^{s+L} \frac{\mathrm{d}s}{\beta}$$

- Average value of β is $\bar{\beta} = R/Q$
- Formal quantities β and α now have pysical meanings, β is the reduced instantaneous betatron wavelength, $\alpha = -\beta'/2$

Phase Space Invariant

- Two-dimensional space (u, p_u) with coordinates u and p_u is called phase space, where p_u is the momentum canonically conjugate to u.
- For constant particle momentum p_u differs from u' only by a constant factor
- Continue working in (u, u')-space and still call it phase space
- (u, u') satisfy the Courant and Snyder invariant

$$E = \pi \frac{u^2 + (\alpha u + \beta u')^2}{\beta}$$

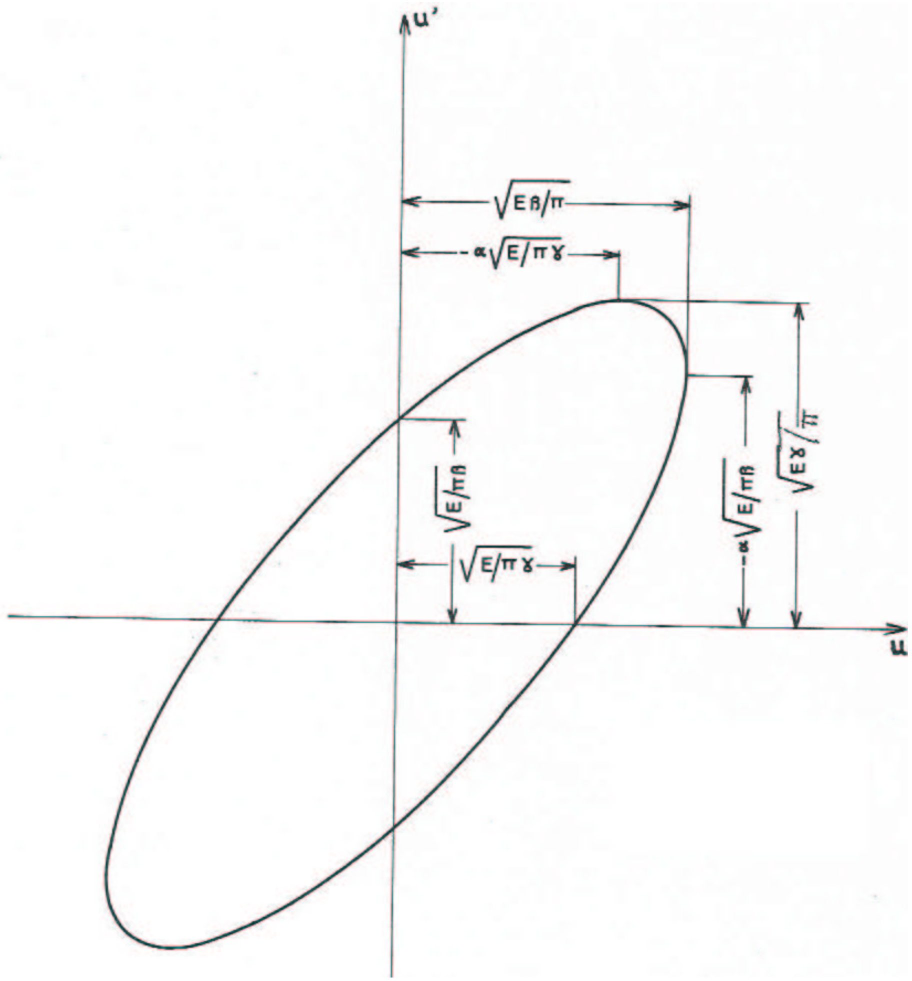
where u, u', α and β are all taken at the same s

- Proof by substitution
- Proof with Wronskian $W = uu_1' - u'u_1$ between (u, u') and a particular solution u_1, using $u_1' = (i - \beta)u_1$

$$W = u_1 \left(\frac{i - \alpha}{\beta} u - u' \right)$$

- Multiplying by the complex conjugate of W and rearranging terms yields the invariant, an ellipse in phase space with parameters related to α, β and γ

Physical Meaning of Ellipse Parameters



- Consider normalised Gaussian density distribution $d(u, u')$ with ellipse in argument of exponential

$$d = \frac{\exp(-\frac{u^2+(\alpha u+\beta u')^2}{2\beta E})}{2\pi E}$$

- RMS radius $\sigma_u^2 = \beta E$
- RMS divergence $\sigma_u'^2 = \gamma E$
- Deduce emittance $E = \sigma_u^2/\beta$ from RMS radius σ_u^2 and β
- Upright ellipses with $\alpha = 0$ are much simpler

Emittance – Acceptance

- Most common definition of emittance E is the area of the ellipse in phase space enclosed by the RMS radius σ_u and RMS divergence σ'_u
- Customary to write e.g. $E = 10\pi$ mm mrad, and not to include the factor π in the number, making it clear that the number is the product of the semi-axes
- Emittance definitions with 2 or 2.5 σ_u also used, often in proton machines
- Width of ellipse in u-direction limited by aperture
- Maximum value of E is called acceptance or admittance
- Only particles with trajectories inside acceptance circulate indefinitely
- Emittance is beam property, acceptance is machine property
- Interest in injecting beam with ellipse shape similar to acceptance ellipse
  - Adapting emittance to acceptance is called matching, cf. examples in later lecture
  - Not to match results in emittance increase, due to processes outside scope of lectures, essentially tune dependence on betatron amplitude and momentum error

Normalised Phase Space

- Often convenient to work in normalised phase space with coordinates (v, v') in which the Courant and Snyder invariant is a circle
- Transformation from (u, u')-space to (v, v')-space achieved by the transformation

$$\begin{pmatrix} v \\ v' \end{pmatrix} = \begin{pmatrix} \beta^{-1/2} & 0 \\ \alpha\beta^{1/2} & \beta^{1/2} \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}$$

  where $v' = \mathrm{d}v/\mathrm{d}\phi$
- The new independent variable $\phi = \int \mathrm{d}s/Q/\beta = \mu/Q$ changes from 0 to 2π around the machine, but it is not the azimuthal angle
- In normalised (v, v')-space the transformation through an element with phase advance μ is represented by counter-clockwise rotation by angle μ
- Applying the transformation to the horizontal equation of motion yields that of a driven harmonic oscillator

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + Q^2 v = \frac{Q^2 \beta^{3/2}}{\rho} \frac{\Delta p}{p}$$

Dispersion I

- Include r.h.s. in horizontal equation of motion and solve it for $\Delta p/p \neq 0$ and piecewise constant $K(s) > 0$ and $\rho(s)$

$$M(s|s_0) = \begin{pmatrix} \cos\varphi & K_x^{-1/2}\sin\varphi & \frac{1-\cos\varphi}{\rho K_x} \\ -K_x^{1/2}\sin\varphi & \cos\varphi & \frac{\sin\varphi}{\rho K_x^{1/2}} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\varphi = K_x^{1/2}(s - s_0)$ and $K_x = \rho^{-2} - K$

- More convenient form for $K_x < 0$

$$M(s|s_0) = \begin{pmatrix} \cosh\psi & (-K_x)^{-1/2}\sin\psi & \frac{\cosh\psi - 1}{\rho(-K_x)} \\ (-K_x)^{1/2}\sinh\psi & \cosh\psi & \frac{\sinh\psi}{\rho(-K_x)^{1/2}} \\ 0 & 0 & 1 \end{pmatrix}$$

with $\psi = (-K_x)^{1/2}(s - s_0)$

- As for betatron oscillations, the matrix for a string of elements is the product of the element matrices

Dispersion II

- Let matrix for a period of lattice be

$$M(s_0 + L|s_0) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

- Dispersion $D(s_0)$ and its derivative $D'(s_0)$ defined as periodic solutions of equation of motion with period L and $\Delta p/p = 1$, and obtained by solving

$$\begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix} = M(s_0 + L|s_0) \begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix}$$

- Integral representation of $D(s)$, obtained by solving normalised equation of motion and transforming it back into (x, x')-space, is often used in further calculations

$$D(s) = \frac{\beta^{1/2}(s)}{2\sin\pi Q} \int_s^{s+C} \frac{\beta^{1/2}(\sigma)[\cos\mu(\sigma) - \cos\mu(s) - \pi Q]\mathrm{d}\sigma}{\rho(\sigma)}$$

Parameters in Longitudinal Dynamics

- Approximate value of the average dispersion $\bar{D}$

$$\bar{D} \approx R/Q^2$$

- Important quantities in longitudinal dynamics
  - Momentum compaction $\alpha_c = (\Delta C/C)/(\Delta p/p) \approx 1/Q^2$ with circumference C
  - Slip factor $\eta = (\Delta T/T)/(\Delta p/p) = \beta_r^2(\Delta T/T)/(\Delta E/E) = \alpha_c - 1/\gamma_r^2$ with transit time T, and β_r, γ_r for reference particle
  - Transition energy with $\eta = 0$ where relativistic factor of reference particle equal to $\gamma_t \approx Q$
- Replacing (x, x') by (D, D') in Courant-Snyder invariant yields

$$\mathcal{H} = \frac{D^2 + (\alpha D + \beta D')^2}{\beta}$$

- $\mathcal{H}$ is a pseudo-invariant that changes only in bending magnets
- $\mathcal{H}$ determines the equilibrium beam size in machines with quantum excitation and synchrotron radiation damping

Conclusions I

- Assembled basic tools for studying lattices to lowest order in linear approximation
- Effect of beam line element described by matrices obeying the rules of matrix algebra
- Standard form of matrices for repeat length with parameters α, β, γ and μ, all with physical interpretation
- Derived parameters to be used in longitudinal dynamics
- Left detailed derivations to tutorials
- Obvious extensions:
  - Errors in alignment, excitation, and field shape $\Rightarrow$ distortion of closed orbit, beating of β-functions, (x, y)-coupling
  - Chromatic effects due to $\Delta p/p \neq 0$
  - Extension to 6×6 matrices in linear approximation and 6D maps including terms of higher than first order
  - Non-linear resonances
  - Dynamic aperture