Accelerator Physics and Neutrino Beams

III. RF Acceleration

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http://keil.home.cern.ch/keil/MuMu/Doc/NuSchool05/talk3.pdf

Programme of Third Lecture – RF Acceleration

- Uses of RF systems
- Recapitulate parameters in longitudinal dynamics
- Synchronous particle
- Non-synchronous particle
- Phase stability
- Hamiltonian formulation
- Oscillation amplitudes
- Limits of stable motion
- Adiabatic damping

Uses of RF Systems

- Accelerate particles in synchrotrons with rising guide field $\dot{B} > 0$, where RF system supplies just that amount of energy that is needed to keep particles near reference orbit
- Compensate synchrotron radiation loss $U_s = 4\pi r_c E_0 \beta^3 \gamma^4 / 3\rho$ of electrons with classical radius r_c and rest mass E_0 in ring with bending radius ρ in an e^+e^- storage ring operating at constant energy or in a synchrotron light source
- Keep beams bunched in storage rings for protons and muons, where bunch length σ_s is limited, by accelerating late particles in bunch, and decelerating early particles in bunch, pushing both towards the bunch centre

Recapitulate Parameters in Longitudinal Dynamics

- Approximate value of the average dispersion $\bar{D}$

$$\bar{D} \approx R/Q^2$$

- Important quantities in longitudinal dynamics
  - Momentum compaction $\alpha_c = (\Delta C/C)/(\Delta p/p) \approx 1/Q^2$ with circumference C
  - Slip factor $\eta = (\Delta T/T)/(\Delta p/p) = \beta_r^2(\Delta T/T)/(\Delta E/E) = \alpha_c - 1/\gamma_r^2$ with transit time T, and β_r, γ_r for reference particle
  - Transition energy with $\eta = 0$ where relativistic factor of reference particle equal to $\gamma_t \approx Q$
- Replacing (x, x') by (D, D') in Courant-Snyder invariant yields

$$\mathcal{H} = \frac{D^2 + (\alpha D + \beta D')^2}{\beta}$$

- $\mathcal{H}$ is a pseudo-invariant that changes only in bending magnets
- $\mathcal{H}$ determines the equilibrium beam size in machines with quantum excitation and synchrotron radiation damping

Synchronous Particle I

- Acceleration V across RF cavity with peak acceleration $\hat{V}$ and circular frequency ω slowly varying with time or constanr

$$V = \hat{V} \int_0^t \omega \mathrm{d}t' = \hat{V} \sin \varphi(t)$$

- ω is integer multiple, harmonic number h, of circular revolution frequency Ω_s of synchronous particle

$$\omega = h\Omega_0 = \frac{h\beta_s c}{R_s}$$

- RF system arranged such that synchronous phase φ_s is constant
- Mean bending field $\overline{B}_s$ and synchronous momentum p_s related by

$$e\overline{B}_s R_s = p_s = m_0 c(\beta\gamma)_s$$

- Rate of momentum change at constant radius

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = eR_s \frac{\mathrm{d}\overline{B}_s}{\mathrm{d}t}$$

Synchronous Particle II

- Momentum change Δp_s in one revolution within time Δt

$$\Delta p_s = eR_s \frac{\mathrm{d}\overline{B}_s}{\mathrm{d}t} \Delta t = \frac{2\pi e R_s^2}{\beta_s c} \frac{\mathrm{d}\overline{B}_s}{\mathrm{d}t}$$

- Use relativistic relation $\Delta E = c\beta\Delta p$, and convert to energy change ΔE_s in a revolution

$$\Delta E_s = c\beta_s \Delta p_s = 2\pi e R_s^2 \frac{\mathrm{d}\overline{B}_s}{\mathrm{d}t}$$

- On the other hand, ΔE_s is given by the RF system

$$\Delta E_s = e\hat{V}\sin\varphi_s$$

- Solve last 2 equations for $\hat{V}$ and conclude that $\hat{V} \propto R_s^2$

$$\hat{V} = \frac{2\pi R_s^2}{\sin\varphi_s} \frac{\mathrm{d}\overline{B}_s}{\mathrm{d}t}$$

- Strong flavour of synchrotron with $\mathrm{d}\overline{B}_s/\mathrm{d}t > 0$
- In e^+e^- storage ring with $U_s > 0$ have simply $\hat{V} = U_s/\sin\varphi_s$

Difference Equations for Non-Synchronous Particles

- Use energy offset ΔE and RF phase φ, counted from previous zero crossing, as canonical variables

$$
\begin{aligned}
\varphi_{n+1} &= \varphi_n + \frac{2\pi h\eta}{\beta_s E_s}\Delta E_{n+1} \\
\Delta E_{n+1} &= \Delta E_n + eV(\sin\varphi_n - \varphi_s)
\end{aligned}
$$

  where n labels the RF cavity traversals, s labels the synchronous particle, h is harmonic number, E is particle energy, eV is peak acceleration in RF cavity with RF waveform $V(t) = V\sin\omega t$

- Linearise for $\Delta\varphi = \varphi - \varphi_s$, and write with matrix M

$$
\begin{pmatrix} \Delta\varphi \\ \delta E \end{pmatrix}_{n+1} = \underbrace{\begin{pmatrix} 1 + \frac{2\pi h\eta eV\cos\varphi_s}{\beta_s^2 E_s} & \frac{2\pi h\eta}{\beta_s^2 E_s} \\ eV\cos\varphi_s & 1 \end{pmatrix}}_{M} \begin{pmatrix} \Delta\varphi \\ \delta E \end{pmatrix}_n
$$

Stability of Longitudinal Motion

- Use machinery of transverse motion and find stability criterion for synchrotron motion

$$0 < -\frac{\pi h \eta e V \cos\varphi_s}{2\beta_s^2 E_s} < 1$$

- Sign of η determines stable quadrant of φ
  - In machine below transition with $\eta < 0$ and $\gamma_r < \gamma_t$, stability needs $\cos\varphi_s > 0$ and $0 < \varphi_s < \pi/2$
  - In machine above transition with $\eta > 0$ and $\gamma_r > \gamma_t$, stability needs $\cos\varphi_s < 0$ and $\pi/2 < \varphi_s < \pi$
  - RF system decelerates for $\pi < \varphi_s < 2\pi$
- Find synchrotron tune Q_s from M for one turn with $2\cos 2\pi Q_s = Tr(M)$

$$Q_s = \left(-\frac{h \eta e V \cos\varphi_s}{2\pi\beta_s^2 E_s}\right)^{1/2}$$

Differential Equations of Motion

- Work to first order in deviations $\Delta\Omega = \Omega - \Omega_s$, $\Delta\varphi = \varphi = \varphi_s$, $\Delta p = p - p_s$, $\Delta E = E - E_s$ from synchronous particle with subscript s
- Distribute RF system in circumference, interpret $\hat{V}$ as circumferential acceleration, and use revolution time τ_s
- Use canonical variables in differential equations, and choose between $(\varphi, \tau_s \Delta E)$ or $(\varphi, \Delta E/\Omega_s)$ with circular revolution frequency $\Omega_s = 1/\tau_s$
- Both products $(\varphi \tau_s \Delta E)$ and $(\varphi \Delta E/\Omega_s)$ have dimension of action
- Get differential equations from difference equations by dividing by τ_s

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}t} &= \frac{2\pi h\eta}{\beta_s^2 \tau_s^2 E_s}(\tau_s \Delta E) \\ \frac{\mathrm{d}(\tau_s \Delta E)}{\mathrm{d}t} &= eV(\sin\varphi - \sin\varphi_s) \end{aligned}$$

- Combine to get second-order ODE for phase with precautions for slowly varying parameters E_s and η

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{E_s}{\eta}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{2\pi h\eta eV(\sin\varphi - \sin\varphi_s)}{\beta_s \tau_s^2}$$

Phase Stability I

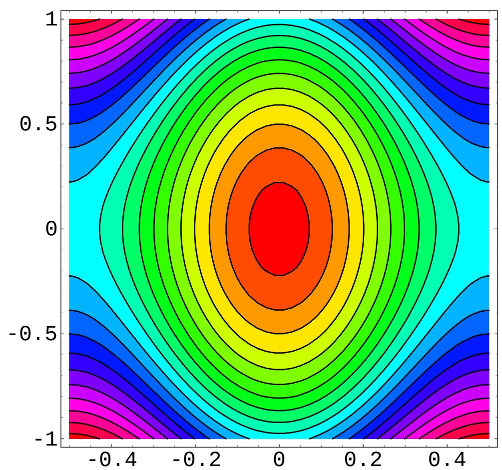
- ODE for φ has first integral of motion for constant E_s and η

$$(\mathrm{d}\varphi/\mathrm{d}t)^2 + 4\pi h\eta\hat{V}(\cos\varphi + \varphi\sin\varphi_s)/\beta_s^2\tau_s^2 E_s$$

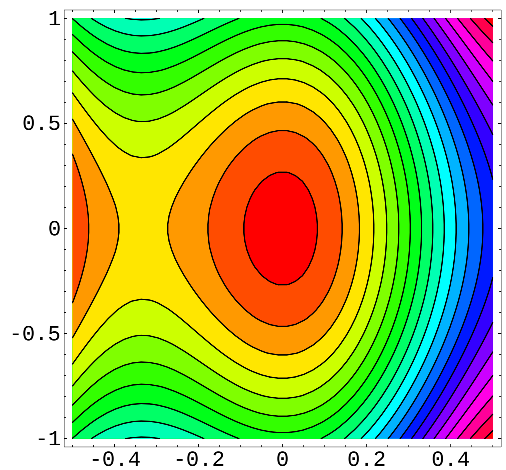
- Guess a Hamiltonian $\mathcal{H}(\varphi, W)$ with $W = \Delta E/\Omega_s$ and $\mathcal{H}(\varphi_s, 0) = 0$

$$\mathcal{H}(\varphi, W) = \frac{h\eta\Omega_s W^2}{2p_s R_0} + \frac{e\hat{V}}{2\pi}\left[\cos\varphi - \cos\varphi_s + (\varphi - \varphi_s)\sin\varphi_s\right]$$

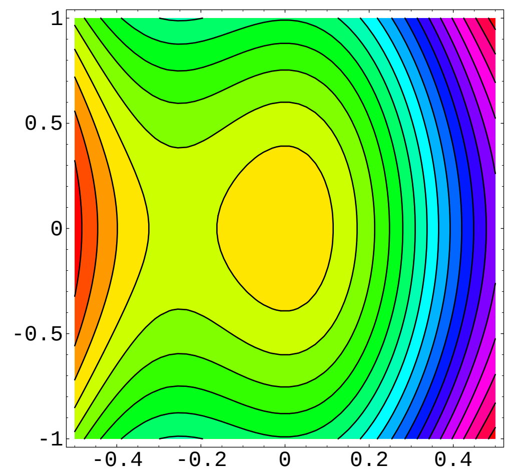
- Phase space trajectories are level lines of $\mathcal{H}$



$\varphi_s = \pi$



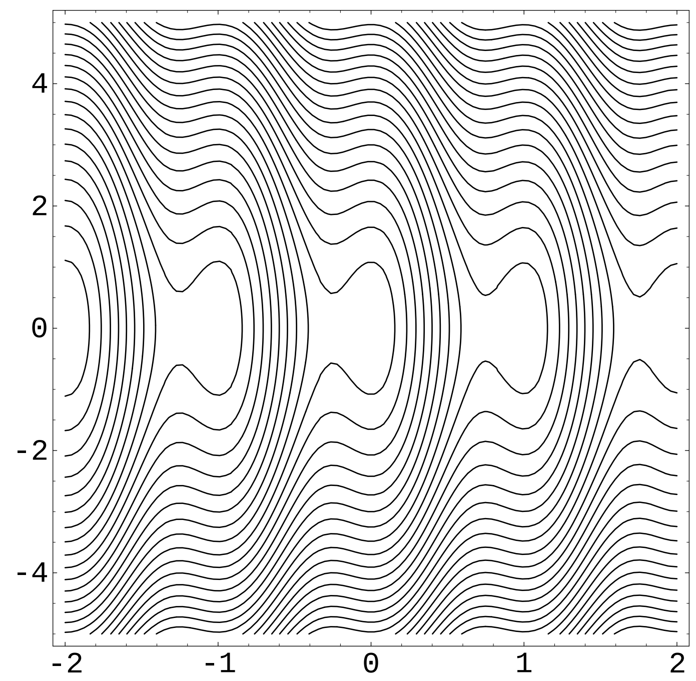
$\varphi_s = 5\pi/6$



$\varphi_s = 3\pi/4$

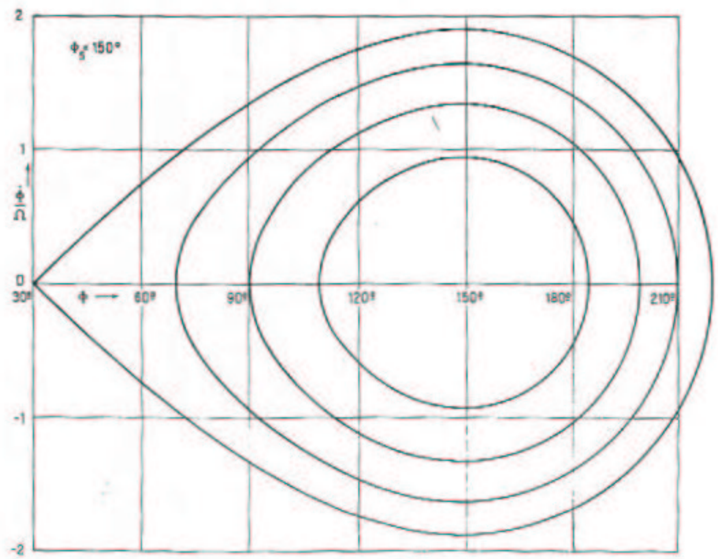
Phase Stability II

- Abscissa is phase in units of 2π
- Stable fixed point at $(n, 0)$ with n integer
- Unstable fixed point at $(n+1/2-2\varphi_s, 0)$
- Closed trajectories around stable fixed points, limited by separatrices
- Area inside separatrices is called bucket
- Width and height of stable trajectories shrink when φ_s decreases
- Outside stationary bucket at $\varphi_s = \pi$ trajectories oscillate in energy and stream in phase
- Outside moving buckets at $\varphi_1 < \pi$ trajectories pass from top to bottom between buckets



$\varphi_s = 3\pi/4$

Trajectories and Buckets



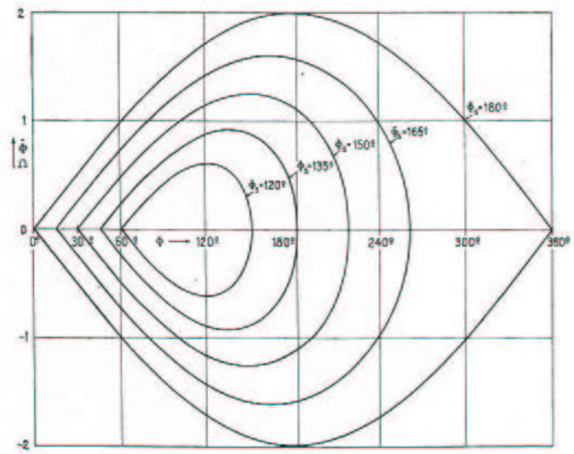


Fig. 35 : Famille de séparatrices

- Trajectories inside moving bucket and separatrix for $\varphi_s = 3\pi/4$
- Separatrices for $2\pi/3 \leq \varphi_s \leq \pi$ in $\pi/12$ steps
- Bucket height, width and area shrink when φ_s decreases from $\varphi_s = \pi$
- Formulae for bucket height, width and area?

Bucket Height

- Use Hamiltonian $\mathcal{H}$ and solve for W

$$\mathcal{H}(\pi - \varphi_s, 0) == \mathcal{H}(\varphi_s, W)$$

- LHS is $\mathcal{H}$ at unstable fixed point
- RHS is $\mathcal{H}$ at phase of stable fixed point
- Solution for half bucket height $\frac{\Delta E_b}{E_s}$

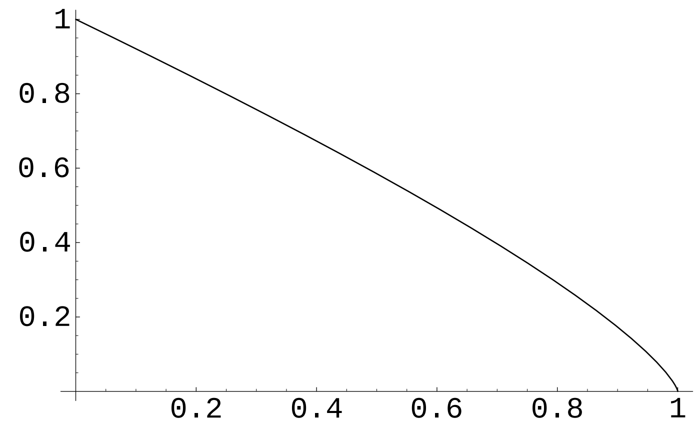
$$\frac{\Delta E_b}{E_s} = \beta_s \left(\frac{2e\hat{V}}{\pi h \eta E_s} \right)^{1/2} B(\varphi_s)$$

with

$$B(\varphi_s) = \sqrt{|(\pi/2 - \varphi_s)\sin\varphi_s - \cos\varphi_s|}$$

and $B(\varphi_s) \to 1$ for $\sin\varphi_s \to 0$

- Convert to momentum with $\Delta p/p = \beta^{-2}\Delta E/E$



$B(\varphi_s)$ vs. $\sin\varphi_s$

Limits on Bucket Width in Phase

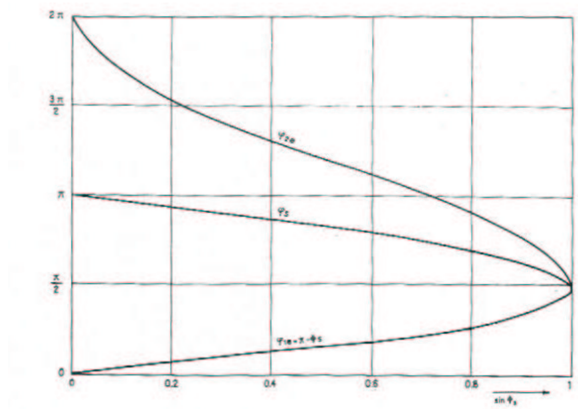
- Left edge of bucket at $\varphi_{1e} = \pi - \varphi_s$
- No closed expression for right edge of bucket at φ_{2e}
- Use Hamiltonian $\mathcal{H}$

$$\mathcal{H}(\pi - \varphi_s, 0) == \mathcal{H}(\varphi_{2e}, 0)$$

- Find transcendental equation for φ_{2e}

$$\cos\varphi_{2e} - \varphi_{2e}\sin\varphi_{2e} = (\pi - \varphi_s)\sin\varphi_s - \cos\varphi_s$$

- No bucket left for $\varphi_s = \pi/2$



Lower edge of bucket φ_{1e}, stable phase angle φ_s and upper edge of bucket φ_{2e} as functions of $\sin\varphi_s$

Bucket Area

- Bucket area defined as area inside separatrix for given φ_s, and known analytically only for stationary buckets with $\Gamma = \sin\varphi_s = 0$

$$A(0) = \frac{8\beta_s}{\pi f_{\mathrm{RF}}} \left(\frac{e\hat{V}E}{2\pi h \eta} \right)^{1/2}$$

- $A(\Gamma) < A(0)$ for $\Gamma > 0$
- $A(\Gamma)$ only known numerically
- $A(\Gamma) \to 0$ for $\Gamma \to 1$
- $A(\Gamma)$ has dimension eVs
- Graph shows $A(\varphi_s)/A(0)$ in first quadrant
- Note different origin of φ: Stationary buckets have $\varphi_s = 0$