# Neutrino Factory Summer School 2005 - Tutorials

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Arguments:

1) Step-by-step calculations of $\nu$ oscillation probabilities of interest for future LBL experiments

2) Elements of statistical analysis

Disclaimer: I have freely elaborated upon existing literature. I'll be happy to provide references upon request, and to correct any error you may find in this manuscript. Just write me a note at the address: eligio.lisi@ba.infn.it ; Your errata-corrige are very important!

# Conversion factors

**Exercise n. 1 :** Prove that $\frac{\Delta m^2 L}{4E} = 1.267 \left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{km}\right)\left(\frac{GeV}{E}\right)$

$$= 1.267 \left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{m}\right)\left(\frac{MeV}{E}\right)$$

$\hbar c = 197.327\ MeV \cdot fm \equiv 1$ in natural units;

Therefore : $1\ MeV \cdot 1\ m = 5.0677 \times 10^{12}$

Then: $\frac{\Delta m^2 L}{4E} = \frac{1}{4}\left(\frac{\Delta m^2}{eV^2}\ eV^2\right)\left(\frac{L}{m} \cdot m\right)\left(\frac{MeV}{E} \cdot \frac{1}{MeV}\right)$

$$= \frac{1}{4}\left(\frac{1\ eV^2 . 1\ m}{1\ MeV}\right)\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{L}{m}\right)\left(\frac{MeV}{E}\right)$$

$$\frac{1}{4}\frac{eV^2 m}{MeV} = \frac{1}{4} \times 10^{-12}\ \frac{MeV^2\, 1\, m}{1\ \cancel{MeV}} = \frac{10^{-12}}{4}(MeV.m)$$

$$= 0.25 \times 10^{-12} \times 5.0677 \times 10^{12} = 1.267$$

✓

**Exercise n.2 :** Prove that $1\ \frac{mol}{cm^3} = 4.267 \times 10^{-9}\ MeV^3$

Reminder: $1\ mol = 6.022 \times 10^{23}$ particles (Avogadro number)

$$1\ \frac{mol}{cm^3} = \frac{6.022 \times 10^{23}}{10^{-6}\ m^3}\left(\frac{MeV^3}{MeV^3}\right) = 6.022 \times 10^{29} \frac{1}{(m \cdot MeV)^3}\ MeV^3 = \frac{6.022 \times 10^{29}}{(5.0677 \times 10^{12})^3}\ MeV^3 = 4.627 \times 10^{-9}\ MeV^3 \checkmark$$

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**Exercise n.3 :** Prove that $\frac{2\sqrt{2}G_F N_e E}{\Delta m^2} = 1.526 \times 10^{-7}\left(\frac{N_e}{mol/cm^3}\right)\left(\frac{E}{MeV}\right)\left(\frac{eV^2}{\Delta m^2}\right)$

Reminder: $G_F = 1.16637 \times 10^{-5}\ GeV^{-2} = 1.16637 \times 10^{-11}\ MeV^{-2}$

$$\frac{2\sqrt{2}G_F N_e E}{\Delta m^2} = 2\sqrt{2}\left(1.16637 \times 10^{-11}\ MeV^{-2}\right)\left(\frac{N_e}{mol/cm^3}\ mol/cm^3\right)\left(\frac{E}{MeV} \cdot MeV\right)\left(\frac{eV^2}{\Delta m^2}\frac{1}{eV^2}\right)$$

$$= 3.299 \times 10^{-11}\ \frac{MeV^{-2}MeV}{eV^2}\ \frac{mol}{cm^3}\left(\frac{N_e}{mol/cm^3}\right)\left(\frac{E}{MeV}\right)\left(\frac{eV^2}{\Delta m^2}\right)$$

$$3.299 \times 10^{-11}\ \frac{MeV^{-2}MeV}{eV^2}\ \frac{mol}{cm^3} = 3.299 \times 10^{-11}\ \frac{10^{12}}{\cancel{MeV^3}} \times 4.627 \times 10^{-9}\ \cancel{MeV^3} = 1.526 \times 10^{-7}$$

# Consistent use of $U$ and $U^*$

- $U$ is introduced in the CC interaction Lagrangian and connects quantized <u>fields</u>:

$$\nu_{\alpha L} = \sum_{i=1}^{3} U_{\alpha i}\, \nu_{iL} \qquad (\alpha = e, \mu, \tau\,; \quad UU^{+} = 1)$$

- Reminder: for a given field $\psi$, it is $\overline{\psi}$ (or $\psi^{+}$) which creates particles from vacuum $|0\rangle$.
Therefore, when fields are used to create one-particle <u>states</u> (kets), one has that:

$$|\nu_\alpha\rangle = \sum_i U^{*}_{\alpha i}\, |\nu_i\rangle \qquad \text{(Particle Data Group convention)}$$

- A generic $|\nu\rangle$ can be decomposed as

$$|\nu\rangle = \sum_i \nu^{i}\, |\nu_i\rangle$$

or $$|\nu\rangle = \sum_\alpha \nu^{\alpha}\, |\nu_\alpha\rangle$$ , where $\nu^{i}, \nu^{\alpha}$ are <u>numbers</u> (components)

namely: $\nu^{\alpha} = \langle \nu_\alpha | \nu \rangle =$ complex number
$\nu^{i} = \langle \nu_i | \nu \rangle =$ " "

These components transform as:

$$\nu^{\alpha} = \sum_i U_{\alpha i}\, \nu^{i}$$

<u>Recap</u> :

$$\nu_{\alpha L} = \sum_i U_{\alpha i}\, \nu_{iL} \qquad \text{(fields)}$$

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle \qquad \text{(states)}$$

$$\nu^\alpha = \sum_i U_{\alpha i}\, \nu^i \qquad \text{(components)}$$

$$U_{\alpha i} = \langle \nu_\alpha | \nu_i \rangle$$

This for neutrinos. For antineutrinos, change $U \to U^*$ everywhere.

Neutrino components are often organized into a (column) vector:

$\begin{pmatrix} \nu^e \\ \nu^\mu \\ \nu^\tau \end{pmatrix}$ ; e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the vector of components of a pure $|\nu_e\rangle$ state, in flavor basis.

# Hamiltonian and change of basis

- Evolution equation: $\hat{H}|\nu\rangle = i\frac{d}{dt}|\nu\rangle$
- $\hat{H}$ = hamiltonian operator
- Decomposition on mass basis:

$$\hat{H} = \sum_{ij} |\nu_j\rangle\langle\nu_j|\,\hat{H}\,|\nu_i\rangle\langle\nu_i| = \sum_{ij} H_{ji}\,|\nu_j\rangle\langle\nu_i|$$

- Decomposition on flavor basis:

$$\hat{H} = \sum_{\alpha\beta} |\nu_\beta\rangle\langle\nu_\beta|\,\hat{H}\,|\nu_\alpha\rangle\langle\nu_\alpha| = \sum_{\alpha\beta} H_{\beta\alpha}\,|\nu_\beta\rangle\langle\nu_\alpha|$$

- Relation among matrix components:

$$H_{ji} = \langle\nu_j|\,\hat{H}\,|\nu_i\rangle \qquad \longrightarrow \qquad H_{\beta\alpha} = \sum_{ij} U_{\beta j}\,H_{ji}\,U^*_{\alpha i}$$

$$H_{\beta\alpha} = \langle\nu_\beta|\,\hat{H}\,|\nu_\alpha\rangle$$

In matrix form: $H_{flavor} = U\,H_{mass}\,U^+$

- Evolution equation in matrix form:

$$\sum_\alpha H_{\beta\alpha} \nu^\alpha = i \frac{d}{dt} \nu^\beta$$

$$\underbrace{[H_{\beta\alpha}]}_{3\times 3 \text{ matrix}} \underbrace{[\nu^\alpha]}_{\text{Vector}} = i \frac{d}{dt} \underbrace{[\nu^\beta]}_{\text{Vector}}$$

(This is what goes in the PC for numerical calculations)

- Analogously in mass basis:

$$\sum_i H_{ji} \nu^i = i \frac{d}{dt} \nu^j$$

$$[H_{ji}][\nu^i] = i \frac{d}{dt} [\nu^j]$$

Reminder: in vacuum, $H_{ji} = \delta_{ij} \left( \frac{m_i^2}{2E} + \text{constant} \right)$

# Evolution operator

The evolution operator $\hat{S}(x,0)$ formally solves the equation $\hat{H}|\nu\rangle = i\frac{d}{dt}|\nu\rangle$ with initial condition $|\nu\rangle = |\nu(0)\rangle$ :

$$|\nu(x)\rangle = \hat{S}(x,0)\,|\nu(0)\rangle$$

In vacuum, it is trivial to obtain the evolution operator since $\hat{H}$ is diagonal and constant:

$$H_{ji} = \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & E_3 \end{pmatrix} \cong p\cdot\mathbf{1} + \frac{1}{2E}\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} = \delta_{ij}\left(p + \frac{m_i^2}{2E}\right)$$

$\hat{S} = e^{-i\hat{H}x}$ ; in matrix components (mass basis):

$$S_{ji} = \langle \nu_j|\,\hat{S}\,|\nu_i\rangle = \delta_{ij}\,\cancel{e^{-ipx}}\,e^{-i\frac{m_i^2}{2E}x}$$

since the exponential of a diagonal matrix is the diagonal of exponentials. The overall common phase $e^{-ipx}$ is irrelevant in oscillations.

In flavor components:

$$\hat{S} = \sum_{\alpha\beta} S_{\beta\alpha} |\nu_\beta\rangle\langle\nu_\alpha| \quad \text{with}$$

$$S_{\beta\alpha} = \langle\nu_\beta|\hat{S}|\nu_\alpha\rangle$$

$$= \sum_{ij} \langle\nu_\beta|\nu_j\rangle\langle\nu_j|\hat{S}|\nu_i\rangle\langle\nu_i|\nu_\alpha\rangle$$

$$= \sum_{ij} U_{\beta j}\, \delta_{ij}\, e^{-i\frac{m_i^2}{2E}x}\, U^*_{\alpha i}$$

$$= \sum_i U^*_{\alpha i} U_{\beta i}\, e^{-i\frac{m_i^2}{2E}x} \qquad \text{(vacuum)}$$

- Amplitude for flavor transitions:

$$A(\nu_\alpha \to \nu_\beta) \overset{\text{def}}{=} \langle\nu_\beta|\hat{S}(x,0)|\nu_\alpha\rangle = S_{\beta\alpha} = \sum_i U^*_{\alpha i} U_{\beta i}\, e^{-i\frac{m_i^2}{2E}x}$$

- Flavor transition probability:

$$P(\nu_\alpha \to \nu_\beta) = |A(\nu_\alpha \to \nu_\beta)|^2 = \left| \sum_i U^*_{\alpha i} U_{\beta i}\, e^{-i\frac{m_i^2}{2E}x} \right|^2 \qquad \text{(vacuum)}$$

**Exercise:** Write the evolution operator in vacuum for $2\nu$ in matrix form (in mass & flavor basis), and find the transition probability.

$$\begin{pmatrix} \nu^e \\ \nu^\mu \end{pmatrix} = U \begin{pmatrix} \nu^1 \\ \nu^2 \end{pmatrix} \quad \text{(components)} \quad \text{with} \quad U = U^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{for } 2\nu. \quad (U^+ = U^T)$$

Mass basis: $H_{mass} = p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2E}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} = \frac{1}{2E}\begin{pmatrix} -\frac{\delta m^2}{2} & 0 \\ 0 & +\frac{\delta m^2}{2} \end{pmatrix} + \text{const} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

where $\delta m^2 = m_2^2 - m_1^2$

Then: $S_{mass} = e^{-iH_{mass} x} = \begin{pmatrix} e^{+i\frac{\delta m^2}{4E}x} & 0 \\ 0 & e^{-i\frac{\delta m^2}{4E}x} \end{pmatrix}$

Flavor basis: $S_{flavor} = U\, S_{mass}\, U^T$

$$= \cos\left(\frac{\delta m^2 x}{4E}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\delta m^2 x}{4E}\right)\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$S_{e\mu} = -i\sin 2\theta \sin\left(\frac{\delta m^2 x}{4E}\right)$

$P_{e\mu} = |S_{e\mu}|^2 = \sin^2 2\theta \sin^2\left(\frac{\delta m^2 x}{4E}\right)$

# Exercise:

Prove that, in vacuum,

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta_{ij}\, x}{4E}\right) - 2 \sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta_{ij}\, x}{2E}\right)$$

where

$$\Delta_{ij} = m_i^2 - m_j^2$$

$$J_{\alpha\beta}^{ij} = U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}$$

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= \left| \sum_i U_{\alpha i}^* U_{\beta i}\, e^{-i \frac{m_i^2}{2E} x} \right|^2 \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\, e^{-i \frac{m_i^2}{2E} x}\, e^{+i \frac{m_j^2}{2E} x} \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left( e^{i \frac{m_j^2 - m_i^2}{2E} x} - 1 + 1 \right) \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left( e^{i \frac{m_j^2 - m_i^2}{2E} x} - 1 \right) + \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \\
&= \left( \sum_{i<j} + \sum_{i>j} \right) U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left( e^{i \frac{m_j^2 - m_i^2}{2E}} - 1 \right) + \sum_i U_{\alpha i}^* U_{\beta i} \sum_j U_{\alpha j} U_{\beta j}^* \\
&= \sum_{i>j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left( e^{i \frac{m_j^2 - m_i^2}{2E}} - 1 \right) + \sum_{i>j} U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \left( e^{-i \frac{m_j^2 - m_i^2}{2E}} - 1 \right) + \delta_{\alpha\beta}\delta_{\alpha\beta} \\
&= \sum_{i>j} \left( U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* + U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \right) \left[ \cos\left( \frac{m_j^2 - m_i^2}{2E} x \right) - 1 \right] \\
&\quad + \sum_{i>j} \left( U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* - U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \right) \left[ i \sin\left( \frac{m_j^2 - m_i^2}{2E} x \right) \right] + \delta_{\alpha\beta}
\end{aligned}
$$

$$= \delta_{\alpha\beta} + \sum_{i>j} 2\,\mathrm{Re}\left(U_{\alpha i}^{*} U_{\beta i} U_{\alpha j} U_{\beta j}^{*}\right) \left[\cos\left(\frac{m_j^2 - m_i^2}{2E} x\right) - 1\right]$$

$$- \sum_{i>j} 2\,\mathrm{Im}\left(U_{\alpha i}^{*} U_{\beta i} U_{\alpha j} U_{\beta j}^{*}\right) \sin\left(\frac{m_j^2 - m_i^2}{2E} x\right)$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \mathrm{Re}\left(U_{\alpha i}^{*} U_{\beta i} U_{\alpha j} U_{\beta j}^{*}\right) \sin^2\left(\frac{m_j^2 - m_i^2}{4E} x\right)$$

$$- 2 \sum_{i>j} \mathrm{Im}\left(U_{\alpha i}^{*} U_{\beta i} U_{\alpha j} U_{\beta j}^{*}\right) \sin\left(\frac{m_j^2 - m_i^2}{2E} x\right)$$

which is the form of $P_{\alpha\beta}$ given in the Particle Data Book (2004)

The desired (equivalent) form is obtained by $i \leftrightarrow j$ :

$$P(\nu_\alpha \to \nu_\beta) = \underbrace{\delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta_{ij}\, x}{4E}\right)}_{P_{CP}}$$

$$\underbrace{- 2 \sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta_{ij}\, x}{2E}\right)}_{P_{\cancel{CP}}}$$

where $\Delta_{ij} = m_i^2 - m_j^2$.

$$J_{\alpha\beta}^{ij} = U_{\alpha i}\, U_{\beta i}^{*}\, U_{\alpha j}^{*}\, U_{\beta j}$$

$P_{CP}$ (first two terms) does not change for $U \to U^*$ (CP-conserving)

$P_{\cancel{CP}}$ (last term) changes sign for $U \to U^*$ (CP-violating)

# Properties of $\mathrm{Im}(J_{\alpha\beta}^{ij})$

- Since $J_{\alpha\beta}^{ij} = U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}$, then $\mathrm{Im}(J_{\alpha\beta}^{ij}) \neq 0$ only for $\alpha \neq \beta$ <u>and</u> $i \neq j$

- Let us define $J \equiv \mathrm{Im}(J_{e\mu}^{12})$

**Exercise :** Prove that, for $\alpha \neq \beta$ & $i \neq j$, $\mathrm{Im}(J_{\alpha\beta}^{ij}) = \pm J$

where one gets a $(+)$ sign if $(\alpha\beta) = (e\mu)$ or $(\mu\tau)$ or $(\tau e)$ (flavor cyclic)

or if $(ij) = (1,2)$ or $(2,3)$ or $(31)$ (generation cyclic)

and a $(-1)$ sign otherwise.

E.g. : $$\mathrm{Im}(J_{e\tau}^{12}) = \mathrm{Im}(U_{e1} U_{\tau 1}^* U_{e2}^* U_{\tau 2}) \overset{\text{unitarity of } U}{=} \mathrm{Im}(U_{e1} U_{e2}^* (-U_{e1}^* U_{e2} - U_{\mu 1}^* U_{\mu 2}))$$
$$= \mathrm{Im}(-U_{e1} U_{e2}^* U_{\mu 1}^* U_{\mu 2}) = -\mathrm{Im}(J_{e\mu}^{12}) = -J$$

E.g. : $$\mathrm{Im}(J_{e\tau}^{21}) = +J = \underset{\substack{\uparrow \\ (e\tau) \\ \text{not cyclic}}}{(-1)} \underset{\substack{\uparrow \\ (21) \\ \text{not cyclic}}}{(-1)} \cdot J$$

- Cyclic rules can be embedded in : $\mathrm{Im}(J_{\alpha\beta}^{ij}) = J \cdot \sum_\gamma \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk}$ where the $\epsilon$ tensors are completely antisymmetric.

# Different ways of writing $P_{CP}$

$P_{CP}$ can be written in several different ways ("sum" or "product" forms). They make use of the following trigonometric identity.

**Exercise:** If $x+y+z=0$
Then $\sin 2x + \sin 2y + \sin 2z = -4\sin x \sin y \sin z$

$$\begin{aligned}
&\sin 2x + \sin 2y + \sin 2z \\
&= \sin 2x + \sin 2y + [-\sin(2x+2y)] \\
&= 2\sin x\cos x + 2\sin y\cos y - 2\sin(x+y)\cos(x+y) \\
&= 2\sin x\cos x + 2\sin y\cos y - 2(\sin x\cos z + \cos x\sin y)(\cos x\cos y - \sin x\sin y) \\
&= 2\sin x\cos x + 2\sin y\cos y - 2\sin x\cos x\cos^2 y - 2\cos^2 x\sin y\cos y \\
&\quad + 2\sin^2 x\sin y\cos y + 2\sin^2 y\sin x\cos x \\
&= 2\sin x\cos x(1-\cos^2 y) + 2\sin y\cos y(1-\cos^2 x) + 2\sin^2 x\sin y\cos y + 2\sin^2 y\sin x\cos x \\
&= 4\sin x\cos x\sin^2 y + 4\sin y\cos y\sin^2 x \\
&= 4\sin x\sin y\sin(x+y) = -4\sin x\sin y\sin z \quad \checkmark
\end{aligned}$$

**Exercise:** Prove that $P_{\not{CP}}$ can be put in the following "product" form:

$$P_{\not{CP}} = 8\,\mathrm{Im}\left(J_{\alpha\beta}^{12}\right) \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij} x}{4E}\right)$$

$$P_{\not{CP}}(\nu_\alpha \to \nu_\beta) = -2 \sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta_{ij}}{2E} x\right)$$

$$= -2\left[\mathrm{Im}\, J_{\alpha\beta}^{12} \sin\left(\frac{\Delta_{12} x}{2E}\right) + \mathrm{Im}\, J_{\alpha\beta}^{23} \sin\left(\frac{\Delta_{23} x}{2E}\right) + \mathrm{Im}\, J_{\alpha\beta}^{13} \sin\left(\frac{\Delta_{13} x}{2E}\right)\right]$$

$$= -2\,\mathrm{Im}\, J_{\alpha\beta}^{12} \left[\sin\left(\frac{\Delta_{12} x}{2E}\right) + \sin\left(\frac{\Delta_{23} x}{2E}\right) - \sin\left(\frac{\Delta_{13} x}{2E}\right)\right]$$

$$= -2\,\mathrm{Im}\, J_{\alpha\beta}^{12} \left[\sin\left(\frac{\Delta_{12} x}{2E}\right) + \sin\left(\frac{\Delta_{23} x}{2E}\right) + \sin\left(\frac{\Delta_{31} x}{2E}\right)\right]$$

$(\Delta_{12} + \Delta_{23} + \Delta_{31} = 0)$

$$= +8\,\mathrm{Im}\, J_{\alpha\beta}^{12} \sin\left(\frac{\Delta_{12} x}{4E}\right) \sin\left(\frac{\Delta_{23} x}{4E}\right) \sin\left(\frac{\Delta_{31} x}{4E}\right)$$

$$= +8\,\mathrm{Im}\, J_{\alpha\beta}^{12} \prod_{(i,j)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij} x}{4E}\right)$$

# Equivalent ways of writing $P_{\alpha\beta}$ in vacuum.

Recap: $P(\nu_\alpha \to \nu_\beta) = P_{CP}(\nu_\alpha \to \nu_\beta) + P_{\cancel{CP}}(\nu_\alpha \to \nu_\beta)$

where $P_{CP} = \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}(J^{ij}_{\alpha\beta}) \sin^2\left(\frac{\Delta_{ij} x}{4E}\right)$

$$P_{\cancel{CP}} = -2 \sum_{i<j} \mathrm{Im}(J^{ij}_{\alpha\beta}) \sin\left(\frac{\Delta_{ij} x}{2E}\right)$$

$$\Delta_{ij} = m_i^2 - m_j^2$$

$$J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}$$

Let us make an overview of different equivalent forms for $P_{CP}$ and $P_{\cancel{CP}}$ :

# $P_{CP}$ :

$$P_{CP} = \delta_{\alpha\beta} - 4 \sum_{i<j} Re\left(J_{\alpha\beta}^{ij}\right) \sin^2\left(\frac{\Delta_{ij} x}{4E}\right)$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} Re\left(J_{\alpha\beta}^{ij}\right) \sin^2\left(\frac{\Delta_{ij} x}{4E}\right)$$

$$= \delta_{\alpha\beta} - 4 \sum_{(i,j)}^{\text{cyclic}} Re\left(J_{\alpha\beta}^{ij}\right) \sin^2\left(\frac{\Delta_{ij} x}{4E}\right)$$

# $P_{\cancel{CP}}$ in "sum" form:

$$P_{\cancel{CP}} = -2 \sum_{i<j} \mathrm{Im}\left(J_{\alpha\beta}^{ij}\right) \sin\left(\frac{\Delta_{ij} x}{2E}\right)$$

$$= -2 \sum_{i>j} \mathrm{Im}\left(J_{\alpha\beta}^{ij}\right) \sin\left(\frac{\Delta_{ij} x}{2E}\right)$$

← get (−1) from J and (−1) from sin by changing $i \leftrightarrow j$

$$= -2\,(\pm J) \sum_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij} x}{2E}\right)$$

← $\pm J$ according to $(\alpha,\beta)$ cyclic or not

# $P_{\not{CP}}$ in "product" form :

$$P_{\not{CP}} = +8 \, \mathrm{Im}\left(J^{12}_{\alpha\beta}\right) \prod_{(i,j)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij}\, x}{4E}\right)$$

$$= +8 \, (\pm J) \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij}\, x}{4E}\right)$$

$\leftarrow$ $\pm J$ according to $(\alpha, \beta)$ cyclic or not

$$= +8 \, J \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta_{ij}\, x}{4E}\right)$$

$$= -8 \, \mathrm{Im}\left(J^{12}_{\alpha\beta}\right) \prod_{i<j} \sin\left(\frac{\Delta_{ij}\, x}{4E}\right)$$

$$= -8 \, (\pm J) \prod_{i<j} \sin\left(\frac{\Delta_{ij}\, x}{4E}\right)$$

**Exercise:** Calculate $P(\nu_\alpha \to \nu_\beta)$ in vacuum, in the limit $\frac{\delta m^2 x}{4E} \to 0 = m_2^2 - m_1^2$
(so-called "one dominant mass scale" approximation)

For $\delta m^2 \to 0$, the $P_{\cancel{CP}}$ term vanishes (evident when written in "product" form).

For $\alpha = \beta$:

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4\,\mathrm{Re}\left(J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha}\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

← here $\Delta m^2 = m_3^2 - m_1^2 = m_3^2 - m_2^2$ for $\delta m^2 = 0$

$$= 1 - 4\left(|U_{\alpha 1}|^2 |U_{\alpha 3}|^2 + |U_{\alpha 2}|^2 |U_{\alpha 3}|^2\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2\left(|U_{\alpha 1}|^2 + |U_{\alpha 2}|^2\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2\left(1 - |U_{\alpha 3}|^2\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

For $\alpha \neq \beta$:

$$P(\nu_\alpha \to \nu_\beta) = -4\,\mathrm{Re}\left(J^{13}_{\alpha\beta} + J^{23}_{\alpha\beta}\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$= -4\,\mathrm{Re}\left(U_{\alpha 1} U^*_{\beta 1} U^*_{\alpha 3} U_{\beta 3} + U_{\alpha 2} U^*_{\beta 2} U^*_{\alpha 3} U_{\beta 3}\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$= -4\,\mathrm{Re}\left(U^*_{\alpha 3} U_{\beta 3} \left(U_{\alpha 1} U^*_{\beta 1} + U_{\alpha 2} U^*_{\beta 2}\right)\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$= -4\,\mathrm{Re}\left(U^*_{\alpha 3} U_{\beta 3} U_{\alpha 3} U^*_{\beta 3}\right) \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

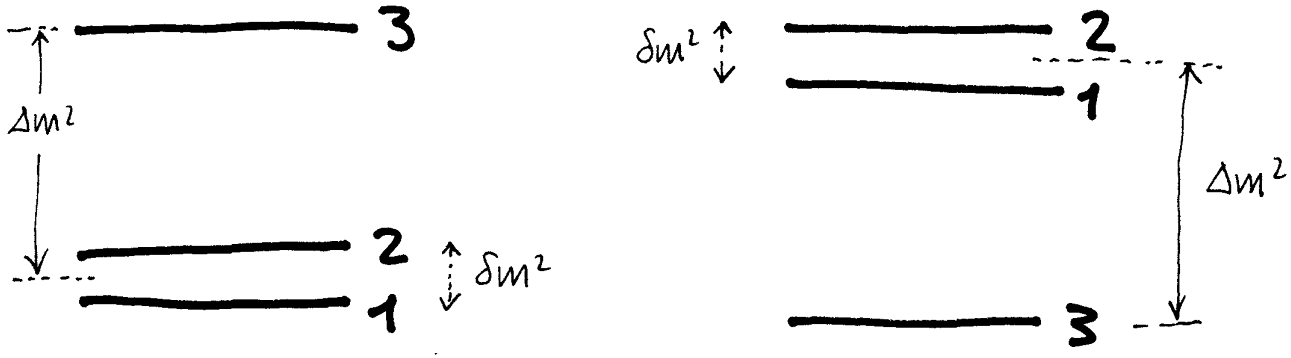
$$= -4|U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

Summarising, at zeroth order in $\frac{\delta m^2 x}{4E}$ :

$$P(\nu_\alpha \to \nu_\beta) = \begin{cases} 1 - 4|U_{\alpha 3}|^2 (1 - |U_{\alpha 3}|^2) \sin^2\left(\frac{\Delta m^2 x}{4E}\right) & (\alpha = \beta) \\ 4|U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) & (\alpha \neq \beta) \end{cases}$$

which is invariant for $\Delta m^2 \to -\Delta m^2$ (no sensitivity to hierarchy) and for $U \Rightarrow U^*$ (no sensitivity to CP violation)

# Note on the mass$^2$ spectrum.




normal hierarchy — inverted hierarchy

Our convention:
$$\begin{cases} \delta m^2 \overset{\text{def}}{=} m_2^2 - m_1^2 \geq 0 \quad \text{always} \\ \Delta m^2 \overset{\text{def}}{=} \left| m_3^2 - \frac{m_1^2+m_2^2}{2} \right| = \left| \frac{\Delta m_{31}^2 + \Delta m_{32}^2}{2} \right| \end{cases}$$

Squared mass matrix $\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$

$$\mathcal{M}^2 = \frac{m_1+m_2}{2}\mathbb{1} + \mathrm{diag}\left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \pm \Delta m^2\right)$$

$+$ : normal
$-$ : inverted

**Exercise :** Calculate $P(\nu_\alpha \to \nu_\alpha)$ in vacuum at 1st order in $\frac{\delta m^2 x}{4E}$

Let us consider normal hierarchy for definiteness:

$$\begin{cases} m_2^2 - m_1^2 = \delta m^2 \\ m_3^2 - m_2^2 = \Delta m^2 - \frac{\delta m^2}{2} \\ m_3^2 - m_1^2 = \Delta m^2 + \frac{\delta m^2}{2} \end{cases}$$

Reminder: $\sin^2(y + \delta y) \simeq \sin^2 y + \sin 2y \cdot \delta y + \mathcal{O}(\delta y^2)$

Then: $\mathrm{Im}\,(J^{ij}_{\alpha\alpha}) = 0$

$\mathrm{Re}\,(J^{ij}_{\alpha\alpha}) = \mathrm{Re}\,(U_{\alpha i} U^*_{\alpha j} U^*_{\alpha i} U_{\alpha j}) = |U_{\alpha i}|^2 |U_{\alpha j}|^2$

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 1}|^2 |U_{\alpha 2}|^2 \sin^2\left(\frac{\delta m^2 x}{4E}\right) - 4|U_{\alpha 2}|^2 |U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 - \frac{\delta m^2}{2}}{4E}\, x\right) - 4|U_{\alpha 1}|^2 |U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 + \frac{\delta m^2}{2}}{4E}\, x\right) \qquad \text{(exact)}$$

At 1st order in $\delta m^2$, the 1st term vanishes and the others become:

$$P_{\alpha\alpha} \simeq 1 - 4|U_{\alpha 2}|^2|U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 4|U_{\alpha 1}|^2|U_{\alpha 3}|^2 \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$
$$-4\left[|U_{\alpha 2}|^2|U_{\alpha 3}|^2\left(\frac{-\frac{\delta m^2}{2}x}{4E}\right) + |U_{\alpha 1}|^2|U_{\alpha 3}|^2\left(\frac{+\frac{\delta m^2}{2}x}{4E}\right)\right]\cdot \sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$
$$- 4|U_{\alpha 3}|^2\left(|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2\right)\left(\frac{\frac{\delta m^2}{2}x}{4E}\right)\sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$$= 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$
$$- 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\frac{|U_{\alpha 1}|^2 - |U_{\alpha 2}|^2}{1-|U_{\alpha 3}|^2}\left(\frac{\frac{\delta m^2}{2}x}{4E}\right)\sin\left(\frac{2\Delta m^2 x}{4E}\right)$$

$\longrightarrow$ next

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\cdot\left[\sin^2\left(\frac{\Delta m^2 x}{4E}\right) + \frac{|U_{\alpha 1}|^2-|U_{\alpha 2}|^2}{|U_{\alpha 1}|^2+|U_{\alpha 2}|^2}\left(\frac{\frac{\delta m^2}{2}x}{4E}\right)\sin\left(\frac{2\Delta m^2 x}{4E}\right)\right]$$

$$\simeq 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\cdot\sin^2\left(\frac{\Delta m^2 + \frac{|U_{\alpha 1}|^2-|U_{\alpha 2}|^2}{|U_{\alpha 1}|^2+|U_{\alpha 2}|^2}\cdot\frac{\delta m^2}{2}}{4E}x\right)$$

which is formally similar to the zeroth order formula, but with

$$\Delta m^2 \to \Delta m^2 + \frac{|U_{\alpha 1}|^2-|U_{\alpha 2}|^2}{|U_{\alpha 1}|^2+|U_{\alpha 2}|^2}\cdot\frac{\delta m^2}{2}$$

A sensitivity to hierarchy arises, since the above equation is not invariant for $\Delta m^2 \to -\Delta m^2$ (the relative sign of the zeroth and first order phases change).

The sensitivity to hierarchy in $P_{\alpha\alpha}$ depends on the amplitude factor $|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)$ and on the ratio : $\dfrac{\frac{\delta m^2}{2}\,\frac{|U_{\alpha 1}|^2-|U_{\alpha 2}|^2}{|U_{\alpha 1}|^2+|U_{\alpha 2}|^2}}{\Delta m^2}$

Phenomenologically, $|U_{e3}|^2 \lesssim$ few % $\rightarrow$ weak sensitivity in $P_{ee}$

Also, although $|U_{\mu 3}|^2 \sim \frac{1}{2}$, it is:

$$\frac{|U_{\mu 1}|^2-|U_{\mu 2}|^2}{|U_{\mu 1}|^2+|U_{\mu 2}|^2} \simeq \frac{\frac{1}{6}-\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} \simeq -\frac{1}{3} \quad \text{and} \quad \frac{\delta m^2}{\Delta m^2} \sim \frac{1}{30}$$

so the ratio is $\sim \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{30} \sim \frac{1}{180} \ll 1$ $\rightarrow$ weak sensitivity in $P_{\mu\mu}$ too.

# Parametrization of $U$.

So far we have used $U$ in generic form. It is useful also to consider $P(\nu_\alpha \to \nu_\beta)$ in specific parametrization. The "standard" (Particle Data Book) parametrization for $U$ is:

$$U = O_{23}\,\Gamma_\delta\, O_{13}\,\Gamma_\delta^{\dagger}\, O_{12} \qquad \text{with } \Gamma_\delta = \mathrm{diag}(1,\ 1,\ e^{i\delta})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \qquad \left(\begin{array}{l}\text{note:} \\ \det U = 1\end{array}\right)$$

**Exercise:**

Prove that $\quad J = \mathrm{Im}\left(J_{e\mu}^{12}\right) = \frac{1}{8}\sin 2\theta_{12}\,\sin 2\theta_{23}\,\sin 2\theta_{13}\,\cos\theta_{13}\,\sin\delta$

**Exercise** : Calculate $P(\nu_e \to \nu_\mu)$ in vacuum at 2nd order in the small parameters $\epsilon = \frac{\delta m^2 x}{4E}$ and $\epsilon' = \sin\theta_{13}$

Note: $\epsilon \ll 1 \Rightarrow \frac{x\,(\mathrm{km})}{E\,(\mathrm{GeV})} \ll 10^4$ ; 2nd order means $O(\epsilon^2, \epsilon'^2, \epsilon\epsilon')$ included.

Let's consider normal hierarchy for definiteness.

$$\underset{\substack{\uparrow \\ \text{exact}}}{P(\nu_e \to \nu_\mu)} = -4 \sum_{(ij)}^{\text{cyclic}} \mathrm{Re}(J^{ij}_{e\mu}) \sin^2\left(\frac{m_i^2 - m_j^2}{4E} x\right) + 8 J \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{m_i^2 - m_j^2}{4E} x\right) \qquad \begin{cases} \epsilon = \frac{\delta m^2 x}{4E} \\ \epsilon' = \sin\theta_{13} \end{cases}$$

Note that $J = \mathrm{Im}(J^{12}_{e\mu}) = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta \sim O(\epsilon')$

Note also that:

- $\mathrm{Re}(J^{12}_{e\mu}) \sin^2\left(\frac{m_1^2 - m_2^2}{4E} x\right)$ ← The 2nd factor is $O(\epsilon^2)$, so the 1st factor can be kept at zeroth order in $\epsilon'$
- $\mathrm{Re}(J^{23}_{e\mu}) \sin^2\left(\frac{m_2^2 - m_3^2}{4E} x\right)$ ← The 1st factor is $O(\epsilon')$, so the 2nd factor can be kept at $O(\epsilon)$
- $\mathrm{Re}(J^{31}_{e\mu}) \sin^2\left(\frac{m_3^2 - m_1^2}{4E} x\right)$ ← The 1st factor is $O(\epsilon')$, so the 2nd factor can be kept at $O(\epsilon)$
- $J . \sin\left(\frac{m_1^2 - m_2^2}{4E} x\right) \sin\left(\frac{m_2^2 - m_3^2}{4E} x\right) \sin\left(\frac{m_3^2 - m_1^2}{4E} x\right)$ ← The 1st & 2nd factors are $O(\epsilon\epsilon')$ altogether, so the other two can be kept at zeroth order in $\epsilon$.

Explicitly, the exact expressions are (for the $J_{e\mu}^{ij}$):

- $J_{e\mu}^{12} = U_{e1} U_{e2}^{*} U_{\mu 1}^{*} U_{\mu 2} = (c_{12} c_{13})(s_{12} c_{13})(-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{-i\delta})(c_{12} c_{13} - s_{12} s_{23} s_{13} e^{i\delta})$

$$= s_{12} c_{12} c_{13}^2 \left(-s_{12} c_{12} c_{23}^2 + s_{12} c_{12} s_{13}^2 s_{23}^2 - c_{12}^2 s_{23} c_{23} s_{13} e^{-i\delta} + s_{12}^2 s_{23} c_{23} s_{13} e^{i\delta}\right)$$

$$\mathrm{Re}\,(J_{e\mu}^{12}) = s_{12} c_{12} c_{13}^2 \left(-s_{12} c_{12} c_{23}^2 + s_{12} c_{12} s_{13}^2 s_{23}^2 + \cos\delta \cdot (-c_{12}^2 s_{23} c_{23} s_{13} + s_{12}^2 s_{23} c_{23} s_{13})\right)$$

$$= s_{12} c_{12} c_{13}^2 \left[ s_{12} c_{12} (-c_{23}^2 + s_{13}^2 s_{23}^2) + \cos\delta\, s_{13} c_{23} s_{23} (s_{12}^2 - c_{12}^2) \right]$$

$$\mathrm{Im}\,(J_{e\mu}^{12}) = s_{12} c_{12} c_{13}^2 \sin\delta \left( s_{23} c_{23} s_{13} c_{12}^2 + s_{12}^2 s_{23} c_{23} s_{13} \right)$$

$$= s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 \sin\delta \equiv J \quad (\text{ok})$$

- $J_{e\mu}^{23} = U_{e2} U_{e3}^{*} U_{\mu 2}^{*} U_{\mu 3} = (s_{12} c_{13})(s_{13} e^{i\delta})(c_{12} c_{23} - s_{12} s_{23} s_{13} e^{-i\delta})(s_{23} c_{13})$

$$= s_{12} s_{13} c_{13}^2 s_{23} \left( c_{12} c_{23} e^{i\delta} - s_{12} s_{23} s_{13} \right)$$

$$\mathrm{Re}\,(J_{e\mu}^{23}) = s_{12} s_{13} c_{13}^2 s_{23} (c_{12} c_{23} \cos\delta - s_{12} s_{23} s_{13}) = s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 \cos\delta - s_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2$$

$$\mathrm{Im}\,(J_{e\mu}^{23}) = s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} \sin\delta = \mathrm{Im}\, J_{e\mu}^{12} \equiv J \quad (\text{ok})$$

- $J_{e\mu}^{31} = U_{e3} U_{e1}^{*} U_{\mu 3}^{*} U_{\mu 1} = (s_{13} e^{-i\delta})(c_{12} c_{13})(s_{23} c_{13})(-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta})$

$$= (c_{12} s_{23} s_{13} c_{13}^2)(-s_{12} c_{23} e^{-i\delta} - c_{12} s_{23} s_{13})$$

$$\mathrm{Re}\,(J_{e\mu}^{31}) = c_{12} s_{23} s_{13} c_{13}^2 (-s_{12} c_{23} \cos\delta - c_{12} s_{23} s_{13}) = - s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 \cos\delta - c_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2$$

$$\mathrm{Im}\,(J_{e\mu}^{31}) = s_{12} c_{12} s_{23} c_{23} s_{13} c_{13}^2 \sin\delta \equiv J \quad (\text{ok})$$

Let's now expand the various pieces as remarked before:

- $\mathrm{Re}\, J_{e\mu}^{12} \simeq -s_{12}^2 c_{12}^2 c_{23}^2$ at zeroth order in $\epsilon'$
- $\sin^2\left(\frac{m_2^2-m_3^2}{4E}x\right) = \sin^2\left(\frac{-\Delta m^2+\delta m^2/2}{4E}x\right) \simeq \sin^2\left(\frac{\Delta m^2 x}{4E}\right) - \left[\sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\right]\left(\frac{\delta m^2 x}{4E}\right)$ at $O(\epsilon)$
- $\sin^2\left(\frac{m_3^2-m_1^2}{4E}x\right) = \sin^2\left(\frac{\Delta m^2+\delta m^2/2}{4E}x\right) \simeq \sin^2\left(\frac{\Delta m^2 x}{4E}\right) + \left[\sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\right]\cdot\left(\frac{\delta m^2 x}{4E}\right)$ at $O(\epsilon)$

and we have used $\sin^2(y+\delta y) \simeq \sin^2 y + 2\sin y\cos y\,\delta y + O(\delta y^2)$

We can now calculate the four pieces of $P_{e\mu}$ at 2nd order in $\epsilon, \epsilon'$:

- $-4\,\mathrm{Re}\,(J_{e\mu}^{12})\sin^2\left(\frac{m_1^2-m_2^2}{4E}x\right) \simeq 4 s_{12}^2 c_{12}^2 c_{23}^2 \left(\frac{\delta m^2 x}{4E}\right)^2 \quad \leftarrow O(\epsilon^2)$
- $-4\,\mathrm{Re}\,(J_{e\mu}^{23})\,\delta m^2\left(\frac{m_2^2-m_3^2}{4E}x\right) \simeq -4\left(s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta - s_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2\right)\times$

$$\times\left(\sin^2\left(\frac{\Delta m^2 x}{4E}\right) - \sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\cdot\left(\frac{\delta m^2 x}{4E}\right)\right)$$

$$\simeq \left(-4 s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta + 4 s_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2\right)\cdot\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$+ 4 s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta\,\sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\left(\frac{\delta m^2 x}{4E}\right) \quad \leftarrow O(\epsilon'^2) + O(\epsilon'\epsilon)$$

- $$-4\,\mathrm{Re}\,J_{e\mu}^{31}\sin^2\left(\frac{m_3^2-m_1^2}{4E}x\right) \simeq -4\left(-s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta - c_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2\right) \times \left(\sin^2\left(\frac{\Delta m^2 x}{4E}\right) + \sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\left(\frac{\delta m^2 x}{4E}\right)\right)$$

$$\simeq \left(4s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta + 4c_{12}^2 s_{13}^2 c_{13}^2 s_{23}^2\right)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$+4s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta\,\sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\left(\frac{\delta m^2 x}{4E}\right) \leftarrow O(\epsilon'^2) + O(\epsilon\epsilon')$$

- $$8J\sin\left(\frac{m_1^2-m_2^2}{4E}x\right)\sin\left(\frac{m_2^2-m_3^2}{4E}x\right)\sin\left(\frac{m_3^2-m_1^2}{4E}x\right)$$

$$\simeq 8J\left(\frac{\delta m^2}{4E}x\right)\sin^2\left(\frac{\Delta m^2 x}{4E}\right) \leftarrow O(\epsilon'\epsilon)$$

- Grouping all 4 pieces together we get:

$$P(\nu_e\to\nu_\mu) \simeq 4s_{12}^2c_{12}^2c_{23}^2\left(\frac{\delta m^2 x}{4E}\right)^2 + 4s_{23}^2 s_{13}^2 c_{13}^2\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$+8s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\cos\delta\,\sin\left(\frac{\Delta m^2 x}{4E}\right)\cos\left(\frac{\Delta m^2 x}{4E}\right)\cdot\left(\frac{\delta m^2 x}{4E}\right)$$

$$+8s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\sin\delta\,\sin\left(\frac{\Delta m^2 x}{4E}\right)\sin\left(\frac{\Delta m^2 x}{4E}\right)\cdot\left(\frac{\delta m^2 x}{4E}\right)$$

Equivalently we can write:

$$P(\nu_e \to \nu_\mu) \simeq s_{23}^2 \cdot \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 x}{4E}\right) \quad \leftarrow \text{"atmospheric" term}$$

$$+ c_{23}^2 \cdot \sin^2 2\theta_{12} \left(\frac{\delta m^2 x}{4E}\right)^2 \quad \leftarrow \text{"solar" term}$$

$$+ \cos\theta_{13} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12} \cos\left(\frac{\Delta m^2 x}{4E} - \delta\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)\left(\frac{\delta m^2 x}{4E}\right) \quad \leftarrow \text{"interference" term}$$

The factor $\cos\theta_{13}$ can be taken here $\sim 1$ within the stated 2nd order approximation for $P_{e\mu}$.

$$P(\nu_e \to \nu_\mu) = P_{atm} + P_{sol} + P_{interf}$$

$P_{sol}$ dominates for $\epsilon' = \sin\theta_{13} \to 0$

$P_{atm}$ dominates for $\epsilon = \frac{\delta m^2 x}{4E} \to 0$

**Exercise** : In the previous expression for $P_{\mu e} = P_{atm} + P_{sol} + P_{interf}$, prove that $|P_{interf}| \leq P_{sol} + P_{atm}$, i.e., the interference term never dominates.

$P_{e\mu}$ can be written (in vacuum) as:

$$P_{e\mu} = |A + B|^2 = |A|^2 + |B|^2 + 2\,\mathrm{Re}\, A B^*$$
$$= P_{sol} + P_{atm} + P_{interf}$$

with $A = c_{23} \sin 2\theta_{12} \left( \frac{\delta m^2 x}{4E} \right)$

$$B = s_{23} \sin 2\theta_{13}\, e^{i\delta} e^{-i \frac{\Delta m^2 x}{4E}} \sin\left( \frac{\Delta m^2 x}{4E} \right)$$

- If $P_{interf} < 0$, then $|P_{interf}| \leq P_{sol} + P_{atm}$, otherwise it would be $|A+B|^2 < 0$
- If $P_{interf} > 0$, then $P_{interf} \leq P_{sol} + P_{atm}$, otherwise it would be $|A-B|^2 < 0$

Therefore, $|P_{interf}| \leq P_{sol} + P_{atm}$ in any case

Finally, let us remind that $P(\nu_e \to \nu_\mu)$ in vacuum has been calculated for neutrinos in normal hierarchy.

For antineutrinos: $U \to U^*$, i.e., $\sin\delta \to -\sin\delta$

For inverted hierarchy: $\Delta m^2 \to -\Delta m^2$

—

Note also that, in vacuum, $P_{\mu e}$ is equivalent to change initial and final states in $P_{e\mu}$, i.e., $x \to -x$. This is also equivalent to $\sin\delta \to -\sin\delta$ (prove it).

# The algebra of matter vs vacuum hamiltonian

- Vacuum, flavor basis:

$$H = U \frac{\mathcal{M}^2}{2E} U^+$$

$$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

$$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^+ O_{12}$$

$$\Gamma_\delta = \mathrm{diag}(1, 1, e^{i\delta})$$

- Matter, flavor basis:

$$\tilde{H} = U \frac{\mathcal{M}^2}{2E} U^+ + V$$

$$V = \mathrm{diag}(v, 0, 0)$$

$v = \sqrt{2}\, G_F N_e$ $\leftarrow$ "interaction energy difference" between $\nu_e$ and $\nu_{\mu,\tau}$ or "neutrino potential".

For antineutrinos: $v \to -v$ $(V \to -V)$ $N_e$ = electron density

- Formal diagonalization in matter:

$$\tilde{H} = U \frac{\mathcal{M}^2}{2E} U^+ + V \stackrel{\text{def}}{=} \tilde{U} \frac{\tilde{\mathcal{M}}^2}{2E} \tilde{U}^+$$

$\tilde{U}$ = "mixing matrix in matter"

$\tilde{\mathcal{M}}^2$ = "effective squared mass matrix in matter" $= \mathrm{diag}(\tilde{m}_1^2, \tilde{m}_2^2, \tilde{m}_3^2)$

with (1,2,3) labeling fixed by the condition $\tilde{m}_i^2 \to m_i^2$ for $N_e \to 0$

- But: exact diagonalization not necessarily transparent or useful ($\dim \tilde{\mathcal{H}} = 3 \to$ cubic roots), and the exact relations

$$\tilde{U} = \tilde{U}(m_i^2, U)$$
$$\tilde{m}_i^2 = \tilde{m}_i^2(m_j^2, U)$$

may be cumbersome.

- There are various ways to circumvent this difficulty:
  - Use some algebraic shortcuts without making approximations;
  - Make some approximations to get "easier" formulae for $\tilde{U}$ and $\tilde{m}_i^2$.

An example of purely algebraic techniques (with no approximation) is the transformation law for $J$ (vacuum) to $\tilde{J}$ (matter):

$$\tilde{J} = J \, \frac{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}{(\tilde{m}_1^2 - \tilde{m}_2^2)(\tilde{m}_2^2 - \tilde{m}_3^2)(\tilde{m}_3^2 - \tilde{m}_1^2)}$$

where $\tilde{J} = \mathrm{Im}\,(\tilde{J}_{e\mu}^{ij})$ in matter.

In the following, we shall prove this identity through the so-called spectral decomposition theorem of linear algebra.

# Spectral decomposition of H (general)

Let us consider the hamiltonian operator in vacuum $\hat{H}$ (with $\hat{H} = \hat{H}^{+}$) with eigenvalues: $\hat{H}|\nu_i\rangle = \lambda_i |\nu_i\rangle$ , $\lambda_i = \frac{m_i^2}{2E}$ .

One can always write: $\hat{H} = \sum_i \lambda_i \hat{X}^i$
where $\hat{X}_i = |\nu_i\rangle\langle\nu_i|$ is the projector operator onto the state $|\nu_i\rangle$.

The spectral decomposition theorem links $\hat{X}^i$ to $\hat{I}, \hat{H}$ and $\lambda_i$ (where $\hat{I}$ = identity) through the equation:

$$\hat{X}^i = \prod_{j \neq i} \frac{\lambda_j \hat{I} - \hat{H}}{\lambda_j - \lambda_i}$$

Proof: $\hat{X}^i |\nu_k\rangle = \prod_{j \neq i} \frac{\lambda_j \hat{I} - \hat{H}}{\lambda_j - \lambda_i} |\nu_k\rangle = \prod_{j \neq i} \frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_i} |\nu_k\rangle = \delta_{ik} |\nu_k\rangle$

thus $\hat{X}^i$ is really the projector onto $|\nu_i\rangle$.

Summarizing: $\hat{H} = \sum_i \lambda_i \hat{X}^i$ ; $\hat{X}^i = \prod_{j \neq i} \frac{\lambda_j \hat{I} - \hat{H}}{\lambda_j - \lambda_i}$ ; $\lambda_i$ = eigenvalue of $\hat{H}$ (vacuum)

Why should we care about this decomposition?

- Notice that:

$$\hat{I} = \sum_i |v_i\rangle\langle v_i| = \sum_i \hat{X}^i$$

$$\hat{H} = \sum_i \lambda_i |v_i\rangle\langle v_i| = \sum_i \lambda_i \hat{X}^i$$

$$\hat{H}^2 = \hat{H}\hat{H} = \Big(\sum_i \lambda_i |v_i\rangle\langle v_i|\Big)\Big(\sum_j \lambda_j |v_j\rangle\langle v_j|\Big)$$

$$= \sum_i \lambda_i^2 |v_i\rangle\langle v_i| = \sum_i \lambda_i^2 \hat{X}^i$$

- In general, for any operator $\hat{G} = g(\hat{H})$ with $g$ power-expandable:

$$\hat{G} = g(\hat{H}) = \sum_i g(\lambda_i)\hat{X}^i$$

- Actually, for $\dim(\hat{H}) = 3$ we only need the powers $\hat{I}, \hat{H}, \hat{H}^2$, since the Cayley-Hamilton theorem states that $\hat{H}$ satisfies its characteristic equation $\prod_i(\lambda_i\hat{I} - \hat{H}) = 0$ $\rightarrow$ therefore $\hat{H}^3$ is a polynomial in $\hat{I}, \hat{H}, \hat{H}^2$ and so any power $\geq 3$: $\hat{H}^n = f(\hat{I}, \hat{H}, \hat{H}^2)$ , $f$ = polynomial; $n \geq 3$.

# Spectral decomposition of H (flavor basis)

Now we want to represent the projection operators $\hat{X}^i$ in flavor basis.
In particular, given $\hat{H}$ in flavor basis as:

$$H = \begin{pmatrix} H_{ee} & H_{e\mu} & H_{e\tau} \\ H_{\mu e} & H_{\mu\mu} & H_{\mu\tau} \\ H_{\tau e} & H_{\tau\mu} & H_{\tau\tau} \end{pmatrix} \quad , \quad H_{\alpha\beta} = H^*_{\beta\alpha} \quad ,$$

we want to prove that:

$$\begin{cases} X^i_{\alpha\alpha} = \dfrac{(\lambda_i - H_{\beta\beta})(\lambda_i - H_{\gamma\gamma}) - |H_{\beta\gamma}|^2}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)} \\[2ex] X^i_{\alpha\beta} = \dfrac{\lambda_i H_{\alpha\beta} + H_{\alpha\gamma} H_{\gamma\beta} - H_{\alpha\beta} H_{\gamma\gamma}}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)} \end{cases} \qquad \begin{matrix} \alpha \neq \beta \neq \gamma \\ i \neq j \neq k \end{matrix}$$

- First we remind that the coefficients of the characteristic equation $\det(H - \lambda I) = 0$ (i.e. $\prod_i (\lambda_i - \lambda) = 0$) are invariant with respect to any change of basis. They are:

$$\lambda_1 \lambda_2 \lambda_3 = \det(H)$$
$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{Tr}(H) = H_{ee} + H_{\mu\mu} + H_{\tau\tau}$$
$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = H_{ee} H_{\tau\tau} + H_{\mu\mu} H_{\tau\tau} + H_{ee} H_{\tau\tau} - |H_{e\tau}|^2 - |H_{e\mu}|^2 - |H_{\mu\tau}|^2$$

Our goal is to express the projector matrices $X^i$ in terms of:

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad , \quad H = \begin{pmatrix} H_{ee} & H_{e\mu} & H_{e\tau} \\ & H_{\mu\mu} & H_{\mu\tau} \\ & & H_{\tau\tau} \end{pmatrix} \quad (H_{\beta\alpha} = H_{\alpha\beta}^*) \quad , \text{ and}$$

$$H^2 = H \cdot H = \begin{pmatrix} H_{ee}^2 + |H_{\mu e}|^2 + |H_{\tau e}|^2 & H_{ee}H_{e\mu} + H_{e\mu}H_{\mu\mu} + H_{e\tau}H_{\tau\mu} & H_{ee}H_{e\tau} + H_{e\mu}H_{\mu\tau} + H_{e\tau}H_{\tau\tau} \\ & |H_{e\mu}|^2 + H_{\mu\mu}^2 + |H_{\tau\mu}|^2 & H_{\mu e}H_{e\tau} + H_{\mu\mu}H_{\mu\tau} + H_{\mu\tau}H_{\tau\tau} \\ & & |H_{\tau e}|^2 + |H_{\mu\tau}|^2 + H_{\tau\tau}^2 \end{pmatrix}$$

Let us start with $X^1$:

$$X^1 = \frac{(\lambda_2 I - H)(\lambda_3 I - H)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \quad ; \text{ the numerator is:}$$

$$(\lambda_2 I - H)(\lambda_3 I - H) = \lambda_2\lambda_3 I - (\lambda_2 + \lambda_3) H + H^2$$

$$= (\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 - \lambda_1\lambda_2 + \lambda_3\lambda_1) I - (\lambda_2 + \lambda_3) H + H^2$$

$$= (\lambda_2\lambda_3 + \lambda_1\lambda_2 + \lambda_3\lambda_1) I - \lambda_1(\lambda_2 + \lambda_3) I - (\lambda_2 + \lambda_3) H + H^2$$

$$= (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) I - (\lambda_2 + \lambda_3)(\lambda_1 I + H) + H^2$$

$$= (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) I - (\mathrm{Tr} H - \lambda_1)(\lambda_1 I + H) + H^2$$

$$= (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) I + H^2 - H\,\mathrm{Tr} H + \lambda_1 (H - I \cdot \mathrm{Tr} H) + \lambda_1^2 I$$

The diagonal element $X_{ee}^{1}$ is given by:

$$(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\,X_{ee}^{1} = (\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) + (H^2)_{ee} - H_{ee}\,(\mathrm{Tr}\,H) + \lambda_1\,(H_{ee}-\mathrm{Tr}\,H) + \lambda_1^2$$

$$= \left(H_{ee}H_{\tau\tau} + H_{\mu\mu}H_{\tau\tau} + H_{ee}H_{\mu\mu} - |H_{e\tau}|^2 - |H_{e\mu}|^2 - |H_{\mu\tau}|^2\right)$$
$$+ H_{ee}^2 + |H_{\mu e}|^2 + |H_{\tau e}|^2 - H_{ee}\,(H_{ee}+H_{\mu\mu}+H_{\tau\tau}) + \lambda_1\,(H_{ee}-H_{ee}-H_{\mu\mu}-H_{\tau\tau}) + \lambda_1^2$$

$$= \lambda_1^2 - \lambda_1\,(H_{\mu\mu}+H_{\tau\tau}) + H_{\mu\mu}H_{\tau\tau} - |H_{\mu\tau}|^2 = (\lambda_1-H_{\mu\mu})(\lambda_1-H_{\tau\tau}) - |H_{\mu\tau}|^2$$

$$\rightarrow \quad X_{ee}^{1} = \frac{(\lambda_1-H_{\mu\mu})(\lambda_1-H_{\tau\tau}) - |H_{\mu\tau}|^2}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}$$

In general: $$X_{\alpha\alpha}^{i} = \frac{(\lambda_i-H_{\beta\beta})(\lambda_i-H_{\gamma\gamma}) - |H_{\beta\gamma}|^2}{(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)} \qquad \begin{matrix} i\neq j\neq k \\ \alpha\neq\beta\neq\gamma \end{matrix}$$

The off-diagonal element $X^1_{e\mu}$ is given by:

$$(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)X^1_{e\mu} = (H^2)_{e\mu} - H_{e\mu}(\mathrm{Tr}H) + \lambda_1 H_{e\mu}$$

$$= H_{ee}H_{e\mu} + H_{e\mu}H_{\mu\mu} + H_{e\tau}H_{\tau\mu} - H_{e\mu}(H_{ee}+H_{\mu\mu}+H_{\tau\tau}) + \lambda_1 H_{e\mu}$$

$$= \lambda_1 H_{e\mu} + H_{e\tau}H_{\tau\mu} - H_{e\mu}H_{\tau\tau}$$

$$= H_{e\mu}(\lambda_1 - H_{\tau\tau}) + H_{e\tau}H_{\tau\mu}$$

In general: $X^i_{\alpha\beta} = \dfrac{H_{\alpha\beta}(\lambda_i - H_{\gamma\gamma}) + H_{\alpha\gamma}H_{\gamma\beta}}{(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)}$  $\qquad \alpha\neq\beta\neq\gamma$, $i\neq j\neq k$

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Recap in flavor basis:

$H = \sum_i \lambda_i X^i \qquad \lambda_i = \dfrac{m_i^2}{2E}$

$$X^i_{\alpha\alpha} = \frac{(\lambda_i - H_{\beta\beta})(\lambda_i - H_{\gamma\gamma}) - |H_{\beta\gamma}|^2}{(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)}$$

$$X^i_{\alpha\beta} = \frac{\lambda_i H_{\alpha\beta} + H_{\alpha\gamma}H_{\gamma\beta} - H_{\alpha\beta}H_{\gamma\gamma}}{(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)}$$

$i\neq j\neq k$
$\alpha\neq\beta\neq\gamma$

# Spectral decomposition and $J_{\alpha\beta}^{ij}$

We have written $\quad H_{\alpha\beta} = \sum_i \lambda_i X_{\alpha\beta}^i$

But we know that $\quad H_{\beta\alpha} = \sum_i U_{\beta i} \lambda_i U_{\alpha i}^*$

$$\rightarrow H_{\alpha\beta} = \sum_i U_{\alpha i} \lambda_i U_{\beta i}^*$$

Therefore, in flavor basis: $\quad X_{\alpha\beta}^i = U_{\alpha i} U_{\beta i}^*$

and: $\quad J_{\alpha\beta}^{ij} = U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}$

$$= X_{\alpha\beta}^i \left(X_{\alpha\beta}^j\right)^*$$

$$= X_{\alpha\beta}^i X_{\beta\alpha}^j$$

# Matter invariants and $J \leftrightarrow \tilde{J}$ relation

In matter: $\tilde{H} = H + \begin{pmatrix} \nu & & \\ & 0 & \\ & & 0 \end{pmatrix}$ (flavor basis)

Therefore : $\begin{cases} \tilde{H}_{\alpha\beta} \equiv H_{\alpha\beta} & \text{for } (\alpha\beta) \neq (ee) \\ \tilde{H}_{ee} = H_{ee} + \nu & \end{cases}$

The above relations allow to connect quantities in vacuum and matter.
In particular, notice that:

$$p \stackrel{def}{=} H_{e\mu} = \tilde{H}_{e\mu} \stackrel{def}{=} \tilde{p}$$

$$q \stackrel{def}{=} H_{e\tau} H_{\tau\mu} - H_{\tau\tau} H_{e\mu} = \tilde{H}_{e\tau} \tilde{H}_{\tau\mu} - \tilde{H}_{\tau\tau} \tilde{H}_{e\mu} \stackrel{def}{=} \tilde{q}$$

Let us apply this invariance to :

$$J^{ij}_{e\mu} = X^i_{e\mu} X^j_{\mu e} \quad \text{with}$$

$$X^i_{e\mu} = \frac{H_{e\mu}\lambda_i + H_{e\tau} H_{\tau\mu} - H_{\tau\tau} H_{e\mu}}{(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)} = \frac{p\lambda_i + q}{(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)}$$

$$X^j_{\mu e} = \frac{p^* \lambda_j + q^*}{(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)}$$

$$J_{e\mu}^{ij} = \frac{p\lambda_i + q}{(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)} \cdot \frac{p^* \lambda_j + q^*}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_j)}$$

$$= \frac{|p|^2 \lambda_i \lambda_j + |q|^2 + q p^* \lambda_j + p q^* \lambda_i}{(\lambda_i - \lambda_j)\left[(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)\right]}$$

$$\mathrm{Im}\left(J_{e\mu}^{ij}\right) = \frac{\mathrm{Im}(p q^*)\,\cancel{(\lambda_i - \lambda_j)}}{\cancel{(\lambda_i - \lambda_j)}\left[(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)\right]}$$

$$= \frac{\mathrm{Im}\left(H_{e\mu} H_{\mu\tau} H_{\tau e} - H_{\tau\tau} |H_{e\mu}|^2\right)}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)}$$

$$= \frac{\mathrm{Im}\left(H_{e\mu} H_{\mu\tau} H_{\tau e}\right)}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)}$$

Now, it is $\mathrm{Im}(p q^*) = \mathrm{Im}(\hat{p} \tilde{q}^*)$ ; therefore, given $\begin{cases} J = \pm \mathrm{Im}(J_{e\mu}^{ij}) \\ \tilde{J} = \pm \mathrm{Im}(\tilde{J}_{e\mu}^{ij}) \end{cases}$

it follows that :

$$\tilde{J} = J \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_2 - \tilde{\lambda}_3)(\tilde{\lambda}_3 - \tilde{\lambda}_1)}$$

$$= \frac{J (m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)}{(\hat{m}_1^2 - \hat{m}_2^2)(\hat{m}_2^2 - \hat{m}_3^2)(\hat{m}_3^2 - \hat{m}_1^2)}$$

# Calculation of $P(\nu_\alpha \to \nu_\beta)$ in matter (general).

- In principle, at any point $x$, we could locally diagonalize the hamiltonian $\tilde{H}(x) = H + V(x)$ in matter as:

  $$\tilde{H}(x) = \tilde{U}(x) \frac{\tilde{\mathcal{M}}^2(x)}{2E} \tilde{U}^+(x) \qquad \text{with } \tilde{\mathcal{M}}^2 = \mathrm{diag}\left(\tilde{m}_1^2(x), \tilde{m}_2^2(x), \tilde{m}_3^2(x)\right)$$

  and write the local evolution operator as:

  $$\tilde{S}(x+dx, x) \simeq e^{-i\tilde{H}dx} = \tilde{U}(x)\, e^{-i\frac{\tilde{m}_i^2(x)}{2E}\cdot dx}\, \tilde{U}^+(x)$$

- The global evolution operator would then be the time-ordered product over all steps $dx$, with $x \in [x_i, x_f]$:

  $$\tilde{S}(x_f, x_i) = \prod \tilde{S}(x+dx, x) = \mathcal{T}\left(e^{-i\int_{x_i}^{x_f} \tilde{H}(x)\,dx}\right)$$

- This would give $P(\nu_\alpha \to \nu_\beta) = |\tilde{S}_{\alpha\beta}|^2$.

- However, this procedure is not easy in general, neither analitically (when an analytical solution exists) nor numerically, since the numerical evolution of (rapidly) oscillating functions may be unstable after many periods, with large errors accumulating after long distances. Approximate or tractable solutions, whenever possible, are welcome.

In the following, we shall derive some (approximate) solutions in matter by using one or more of the following "tricks":

1) Constant-density approximation
2) Reduction to $2\nu$ evolution
3) Expansion in small parameters
4) Reduction to a suitable basis
5) Use of symmetries in specific parametrizations

Everywhere, '$\sim$' distinguishes quantities in matter.

# Formal solution for $P_{\alpha\beta}$ at constant density

For constant density, we can always write formally

$$\tilde{H} = \tilde{U} \frac{\tilde{\mathcal{M}}^2}{2E} \tilde{U}^+ \quad \text{with no dependence on } x.$$

Then, just as in the vacuum case (where $H = U \frac{\mathcal{M}^2}{2E} U^+$) :

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i<j} Re(\tilde{J}^{ij}_{\alpha\beta}) \sin^2\left(\frac{\tilde{\Delta}_{ij}\, x}{4E}\right)$$
$$- 2 \sum_{i<j} Im(\tilde{J}^{ij}_{\alpha\beta}) \sin\left(\frac{\tilde{\Delta}_{ij}\, x}{2E}\right)$$

with $\tilde{J}^{ij}_{\alpha\beta} = \tilde{U}_{\alpha i} \tilde{U}^*_{\beta i} \tilde{U}^*_{\alpha j} \tilde{U}_{\beta j}$

$\tilde{\Delta}_{ij} = \tilde{m}^2_i - \tilde{m}^2_j$

# $P_{\alpha\beta}$ for constant density and 2 neutrinos only.

- Let us consider the $2\nu$ case in flavor basis,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad , \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = U^*$$

$$\tilde{H} = U \begin{pmatrix} \frac{m_1^2}{2E} & \\ & \frac{m_2^2}{2E} \end{pmatrix} U^T + \begin{pmatrix} V & \\ & 0 \; 0 \end{pmatrix}$$

$$V = \sqrt{2}\, G_F N_e \quad , \quad N_e = \text{const}$$

$$A = 2EV = 2\sqrt{2}\, G_F N_e$$

$$= \frac{1}{2E} \left[ \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right]$$

- It is convenient to put $\tilde{H}$ in traceless form:

$$\tilde{H} = \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta\, \Delta & \sin 2\theta\, \Delta \\ \sin 2\theta\, \Delta & -A + \cos 2\theta\, \Delta \end{bmatrix} \qquad \text{where} \quad \Delta = m_2^2 - m_1^2$$

- Eigenvalues: $\pm \frac{\tilde{\Delta}}{4E}$ , where $\tilde{\Delta} = \Delta \sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}$

- Diagonalizing rotation:

$$\tilde{H} = \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix} \begin{pmatrix} -\frac{\tilde{\Delta}}{4E} & 0 \\ 0 & +\frac{\tilde{\Delta}}{4E} \end{pmatrix} \begin{pmatrix} \cos\tilde{\theta} & -\sin\tilde{\theta} \\ \sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}$$

with $\sin 2\tilde{\theta} = \dfrac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}}$ , $\cos 2\tilde{\theta} = \dfrac{\cos 2\theta - A/\Delta}{\sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}}$

Note that $\tilde{\Delta} \sin 2\tilde{\theta} = \Delta \sin 2\theta$

- Evolution operator in matter:

$$\tilde{S} = e^{-i\hat{H}x} = \begin{pmatrix} c\tilde{\theta} & s\tilde{\theta} \\ -s\tilde{\theta} & c\tilde{\theta} \end{pmatrix} e^{-i\left(-\frac{\tilde{\Delta}}{4E} + \frac{\tilde{\Delta}}{4E}\right)x} \begin{pmatrix} c\tilde{\theta} & -s\tilde{\theta} \\ s\tilde{\theta} & c\tilde{\theta} \end{pmatrix}$$

$$= \cos\left(\frac{\tilde{\Delta}}{4E}x\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\left(\frac{\tilde{\Delta}x}{4E}\right) \begin{pmatrix} -\cos 2\tilde{\theta} & \sin 2\tilde{\theta} \\ \sin 2\tilde{\theta} & \cos 2\tilde{\theta} \end{pmatrix}$$

- Off-diagonal element: $\tilde{S}_{e\mu} = -i \sin\left(\frac{\tilde{\Delta}x}{4E}\right) \sin 2\tilde{\theta}$

- Transition probability: $P_{e\mu} = |\tilde{S}_{e\mu}|^2 = \sin^2 2\tilde{\theta} \sin^2\left(\frac{\tilde{\Delta}x}{4E}\right)$

(as in vacuum, but with $\theta \to \tilde{\theta}$ and $\Delta \to \tilde{\Delta}$)

# Reduction to a "simpler" basis

- Let us consider the general $3\nu$ case in matter with any $N_e(x)$ profile:

$$\tilde{H} = U \frac{\mathcal{M}^2}{2E} U^+ + V$$

$$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

$$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^+ O_{12}$$

$$\Gamma_\delta = \mathrm{diag}(1, 1, e^{i\delta})$$

$$V = \mathrm{diag}(\sqrt{2} G_F N_e, 0, 0) \qquad N_e = N_e(x)$$

<u>Exercise</u> (very simple): prove that
$$\begin{cases} (O_{23}\Gamma_\delta)^+ V (O_{23}\Gamma_\delta) = V \\ \Gamma_\delta^+ O_{12} \mathcal{M}^2 O_{12}^T \Gamma_\delta = O_{12} \mathcal{M}^2 O_{12}^T \end{cases}$$

- Let's go from the flavor basis to a new "primed flavor" basis defined as:

$$\begin{bmatrix} (\nu_e)' \\ (\nu_\mu)' \\ (\nu_\tau)' \end{bmatrix} = (O_{23}\Gamma_\delta)^+ \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} \quad , \quad O_{23}\Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}\, e^{i\delta} \end{pmatrix}$$

- In the primed basis, the hamiltonian is:

$$\tilde{H}' = (O_{23}\Gamma_\delta)^+ H (O_{23}\Gamma_\delta) = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$$

- Properties of $\tilde{H}'$ : the new hamiltonian $\tilde{H}'$ is simpler than $\tilde{H}$, since $\tilde{H}'$ does not depend on $\delta$ and is thus real symmetric. Moreover, it does not depend on $\theta_{23}$.

- Therefore, it is convenient to find first the evolution operator $\tilde{S}'$ in the primed basis, and then the evolution operator $\tilde{S}$ in the true flavor basis as:

$$\tilde{S}(x_f, x_i) = (O_{23}\Gamma_\delta)\,\tilde{S}'(x_f, x_i)\,(O_{23}\Gamma_\delta)^+$$

- In terms of components,

if $\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & \tilde{S}'_{e\mu} & \tilde{S}'_{e\tau} \\ \tilde{S}'_{\mu e} & \tilde{S}'_{\mu\mu} & \tilde{S}'_{\mu\tau} \\ \tilde{S}'_{\tau e} & \tilde{S}'_{\tau\mu} & \tilde{S}'_{\tau\tau} \end{pmatrix}$ then $\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}e^{i\delta} \end{pmatrix} \cdot \tilde{S}' \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23}e^{i\delta} & c_{23}e^{-i\delta} \end{pmatrix}$

$$\rightarrow \begin{cases} \tilde{S}_{ee} = \tilde{S}'_{ee} \\ \tilde{S}_{\mu e} = \tilde{S}'_{\mu e}c_{23} + \tilde{S}'_{\tau e}s_{23}e^{i\delta} \\ \tilde{S}_{\tau e} = -\tilde{S}'_{\mu e}s_{23} + \tilde{S}'_{\tau e}c_{23}e^{i\delta} \\ \tilde{S}_{\mu\mu} = \tilde{S}'_{\mu\mu}c_{23}^2 + \tilde{S}'_{\mu\tau}c_{23}s_{23}e^{-i\delta} + \tilde{S}'_{\tau\mu}c_{23}s_{23}e^{i\delta} + s'_{\tau\tau}s_{23}^2 \\ \tilde{S}_{\tau\mu} = -\tilde{S}'_{\mu\mu}c_{23}s_{23} - \tilde{S}'_{\mu\tau}\tilde{s}_{23}^2 e^{-i\delta} + \tilde{S}'_{\tau\mu}c_{23}^2 e^{i\delta} + \tilde{S}'_{\tau\tau}c_{23}s_{23} \\ \tilde{S}_{\tau\tau} = \tilde{S}'_{\mu\mu}s_{23}^2 - \tilde{S}'_{\mu\tau}c_{23}s_{23}e^{-i\delta} - \tilde{S}'_{\tau\mu}c_{23}s_{23}e^{i\delta} + \tilde{S}'_{\tau\tau}c_{23}^2 \end{cases}$$

(with $\tilde{S}_{e\mu}$, $\tilde{S}_{e\tau}$, $\tilde{S}_{\mu\tau}$ obtained from $\tilde{S}'_{\alpha\beta} \leftrightarrow \tilde{S}'_{\beta\alpha}$ and $+\delta \leftrightarrow -\delta$)

Remark on (a)symmetric density profiles $N_e(x)$

Note that, in general, $\tilde{S}'_{\alpha\beta} \neq \tilde{S}'_{\beta\alpha}$, even if $\tilde{H}'_{\alpha\beta} = \tilde{H}'_{\beta\alpha}$ (real symmetric). Why?

For a generic profile divided into N steps $\{\Delta x_i\}_{i=1,\dots,N}$ and $\sim$constant density (and hamiltonian) in each step:

$$\tilde{S}' = e^{-i\tilde{H}'_N \Delta x_N} e^{-i\tilde{H}_{N-1} \Delta x_{N-1}} \dots e^{-i\tilde{H}'_2 \Delta x_2} e^{-i\tilde{H}'_1 \Delta x_1}$$

Although $(\tilde{H}'_i)^T = \tilde{H}'_i$, the transpose of $\tilde{S}'$ is not equal to $\tilde{S}'$, since the ordering of the steps is reversed from $1,\dots,N$ to $N,\dots,1$ ("reverse" profile):

$$(\tilde{S}')^T = e^{-i\tilde{H}'_1 \Delta x_1} \dots e^{-i\tilde{H}'_N \Delta x_N}$$

Therefore, although: $\tilde{S}'_{\alpha\beta}$ [direct profile] $= \tilde{S}'_{\beta\alpha}$ [reverse profile]

in general one has that: $\tilde{S}'_{\alpha\beta}$ [direct profile] $\neq \tilde{S}'_{\alpha\beta}$ [reverse profile]

unless the direct and reverse profiles are symmetrical (i.e. equal to each other):

$$\tilde{S}'_{\alpha\beta}\,[\text{symmetric}] = \tilde{S}'_{\beta\alpha}\,[\text{symmetric}]$$

In particular, for constant density, $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$.

# Reduction of $P_{\alpha\beta}$ to $(P_{e\mu}, P_{\mu\tau})$ by symmetry

Let us remind that $P(\nu_\alpha \to \nu_\beta) = |\tilde{S}_{\alpha\beta}|^2 = P_{\alpha\beta}$, where $\tilde{S}$ is derived from the simpler $\tilde{S}'$ in the primed basis, as shown before.

Notice that, for a generic profile of $N_e$ :

$$\left\{\begin{aligned}
\tilde{S}_{\mu e} &= \tilde{S}'_{\mu e}\, c_{23} + \tilde{S}'_{\tau e}\, s_{23}\, e^{i\delta} \\
\tilde{S}_{\tau e} &= -\tilde{S}'_{\mu e}\, s_{23} + \tilde{S}'_{\tau e}\, c_{23}\, e^{i\delta} = \pm \tilde{S}_{\mu e}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \\
&\text{Therefore: } P_{\mu e} = P_{\tau e}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{\tau e}
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\tilde{S}_{\tau\mu} &= -\tilde{S}'_{\mu\mu}\, c_{23} s_{23} - \tilde{S}'_{\mu\tau}\, s_{23}^2\, e^{-i\delta} + \tilde{S}'_{\tau\mu}\, c_{23}^2\, e^{i\delta} + \tilde{S}'_{\tau\tau}\, c_{23} s_{23} \\
\tilde{S}_{\mu\tau} &= -\tilde{S}'_{\mu\mu}\, c_{23} s_{23} - \tilde{S}'_{\tau\mu}\, s_{23}^2\, e^{i\delta} + \tilde{S}'_{\mu\tau}\, c_{23}^2\, e^{-i\delta} + \tilde{S}'_{\tau\tau}\, c_{23} s_{23} = \mp S_{\tau\mu}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \\
&\text{Therefore: } P_{\tau\mu} = P_{\mu\tau}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{\mu\tau}
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\tilde{S}_{\tau\tau} &= \tilde{S}'_{\mu\mu}\, s_{23}^2 - \tilde{S}'_{\mu\tau}\, c_{23} s_{23}\, e^{-i\delta} - \tilde{S}'_{\tau\mu}\, c_{23} s_{23}\, e^{i\delta} + \tilde{S}'_{\tau\tau}\, c_{23}^2 \\
\tilde{S}_{\mu\mu} &= \tilde{S}'_{\mu\mu}\, c_{23}^2 + \tilde{S}'_{\mu\tau}\, c_{23} s_{23}\, e^{-i\delta} + \tilde{S}'_{\tau\mu}\, c_{23} s_{23}\, e^{i\delta} + \tilde{S}'_{\tau\tau}\, s_{23}^2 = \pm S_{\tau\tau}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \\
&\text{Therefore: } P_{\tau\tau} = P_{\mu\mu}\Big|_{\substack{s_{23}\to \pm c_{23} \\ c_{23}\to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{\tau\tau}
\end{aligned}\right.$$

The previous relations, together with unitarity of $P_{\alpha\beta}$, allow to express all the probabilities in terms of just two, e.g., $P_{e\mu}$ and $P_{\mu\tau}$, and their transformed :

$$P'_{e\mu} = P_{e\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \qquad P'_{\mu\tau} = P_{\mu\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}}$$

Explicitly :

$$\begin{cases} P_{ee} = 1 - P_{e\mu} - P_{e\tau} = 1 - P_{e\mu} - P'_{e\mu} \\ P_{e\tau} = P'_{e\mu} \\ P_{\mu e} = 1 - P_{\mu\mu} - P_{\mu\tau} = 1 - P_{\mu\mu} - P_{e\mu} + P_{e\mu} - P_{\mu\tau} = P_{e\mu} + P_{\tau\mu} - P_{\mu\tau} = P_{e\mu} + P'_{\mu\tau} - P_{\mu\tau} \\ P_{\mu\mu} = 1 - P_{e\mu} - P_{\tau\mu} = 1 - P_{e\mu} - P'_{\mu\tau} \\ P_{\tau\mu} = P'_{\mu\tau} \\ P_{\tau\tau} = 1 - P_{e\tau} - P_{\mu\tau} = 1 - P'_{e\mu} - P_{\mu\tau} \end{cases}$$

---

Note: it is equivalen to transform either $\begin{pmatrix} s_{23} \to +c_{23} \\ c_{23} \to -s_{23} \end{pmatrix}$ or $\begin{pmatrix} s_{23} \to -c_{23} \\ c_{23} \to +s_{23} \end{pmatrix}$.

# Calculation of $P_{e\mu}$ and $P_{e\tau}$ in matter, at 2nd order in the small parameters $\delta m^2$ and $\sin\theta_{13}$

- The probabilities $P(\nu_e \to \nu_\mu)$ ("golden channel") and $P(\nu_e \to \nu_\tau)$ ("silver channel") are particularly important in the context of future long-baseline experiments.

- We want to show that, for constant density $N_e$, and at 2nd order in the small parameters $\delta m^2$ and $\sin\theta_{13}$, one has that:

$$P(\nu_e \to \nu_\mu) \simeq X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z$$

with

$$\begin{cases} X = \sin^2\theta_{23} \left(\dfrac{\Delta m^2}{A - \Delta m^2}\right)^2 \sin^2\left(\dfrac{A - \Delta m^2}{4E} x\right) & (A = 2\sqrt{2}\, G_F N_e E) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\dfrac{\delta m^2}{A}\right)\left(\dfrac{\Delta m^2}{A - \Delta m^2}\right) \sin\left(\dfrac{Ax}{4E}\right) \sin\left(\dfrac{A - \Delta m^2}{4E} x\right) \\ Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\dfrac{\delta m^2}{A}\right)^2 \sin^2\left(\dfrac{Ax}{4E}\right) \end{cases}$$

- Note: sometimes one finds a further $\cos\theta_{13}$ factor in $Y$. This is, however, irrelevant at the stated 2nd order approximation.

- Once the "golden channel" probability $P(\nu_e \to \nu_\mu)$ is obtained, the "silver channel" probability $P(\nu_e \to \nu_\tau)$ is given by the previous reduction formulae as:

$$P(\nu_e \to \nu_\tau) = P(\nu_e \to \nu_\mu)\Big|_{\substack{c_{23} \to \mp s_{23} \\ s_{23} \to \pm c_{23}}}$$

$$= X^\tau \sin^2 2\theta_{13} - Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z^\tau$$

where: $$\begin{cases} X^\tau = \cos^2\theta_{23} \left(\dfrac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\dfrac{A - \Delta m^2}{4E} x\right) \\ Y = \text{as before} \\ Z^\tau = \sin^2\theta_{23} \sin^2 2\theta_{12} \left(\dfrac{\delta m^2}{A}\right)^2 \sin^2\left(\dfrac{Ax}{4E}\right) \end{cases}$$

---

For antineutrinos: $A \to -A$
$\delta \to -\delta$

---

For inverse hierarchy: $\Delta m^2 \to -\Delta m^2$

- We remind that: $P_{e\mu} = |\tilde{S}_{e\mu}|^2$ with $\tilde{S}_{e\mu} = \tilde{S}'_{e\mu} c_{23} + \tilde{S}'_{e\tau} s_{23} e^{-i\delta}$

Therefore, we only need to calculate $\tilde{S}'_{e\mu}$ and $\tilde{S}'_{e\tau}$ in the primed basis.
Since we have assumed constant density, it is

$$\tilde{S}'_{e\mu} = \tilde{S}'_{\mu e} \quad \text{and} \quad \tilde{S}'_{e\tau} = \tilde{S}'_{\tau e}.$$

Explicitly we have:

$$P_{e\mu} = |\tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{-i\delta}|^2 = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$$

with
$$\begin{cases} A_{e\mu} = 2\,\mathrm{Re}[S'^{*}_{\mu e} S'_{\tau e}]\, c_{23} s_{23} \\ B_{e\mu} = -2\,\mathrm{Im}[S'^{*}_{\mu e} S'_{\tau e}]\, c_{23} s_{23} \\ C_{e\mu} = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}|\, s_{23}^2 \end{cases}$$

- The next "trick" is to reduce the evolution from $3\nu$ to approximately $(2\nu) \oplus (1\nu)$, by making use of an expansion in two phenomenologically small parameters:

$$s_{13}$$

$$\text{and} \quad \frac{\delta m^2}{\Delta m^2}$$

- In the following, a term $T$ will be called of "1st order" if it's proportional to $s_{13}$ or $\delta m^2$:

$$T \sim O_1 \quad \text{if} \quad T \propto s_{13} \quad \text{or} \quad T \propto \delta m^2$$

Analogously,

$$T \sim O_2 \quad \text{if} \quad T \propto \begin{cases} s_{13}^2 \text{ or} \\ (\delta m^2)^2 \text{ or} \\ s_{13} \cdot \delta m^2 \end{cases}$$

etc...

- We shall show that: $\tilde{S}'_{e\mu} \sim O_1$ and $\tilde{S}'_{e\tau} \sim O_1$
Therefore, since $P_{e\mu}$ is a quadratic form in $\tilde{S}'_{e\mu}$ and $\tilde{S}'_{e\tau}$, $P_{e\mu} \sim O_2$ as desired.

- Let's remind that, in the primed basis, it is:

$$\begin{cases} \tilde{H}' = O_{13}\, O_{12}\, \dfrac{\mathcal{M}^2}{2E} (O_{13}\, O_{12})^T + V \\ \mathcal{M}^2 = \mathrm{diag}\left(-\dfrac{\delta m^2}{2}, +\dfrac{\delta m^2}{2}, \Delta m^2\right) \\ V = \mathrm{diag}\left(\sqrt{2}\, G_F N_e, 0, 0\right) \end{cases}$$

← in normal hierarchy and up to a term $\propto \mathbb{1}$

- In the primed basis, the evolution decouples as $3\nu = (2\nu) \oplus (1\nu)$ in two limits:

$$s_{13} \to 0 \quad \Rightarrow \quad O_{13} = \mathbb{1}$$
$$\delta m^2 \to 0 \quad \Rightarrow \quad O_{12}\, \mathcal{M}^2 O_{12}^T = \mathcal{M}^2$$

- It is convenient to define

$$\tilde{H}^l = \lim_{s_{13} \to 0} \tilde{H}'$$
$$\tilde{H}^h = \lim_{\delta m^2 \to 0} \tilde{H}'$$

and to study the evolution operator components $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$ in $\tilde{H}^l$ and $\tilde{H}^h$. The task is particularly simple since both $\tilde{H}^l$ and $\tilde{H}^h$ have only one nontrivial $2\times2$ block submatrix, and we have already completely solved the $2\times2$ case.

<u>Limit $s_{13} \to 0$ in primed basis</u>

$$\tilde{H}^{\ell} = \lim_{s_{13}\to 0} \tilde{H}' = \frac{1}{2E}\left[ O_{12} \begin{pmatrix} -\frac{\delta m^2}{2} & & \\ & +\frac{\delta m^2}{2} & \\ & & \Delta m^2 \end{pmatrix} O_{12}^T + \begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$

$$= \frac{A}{4E}\mathbb{1} + \frac{1}{2E}\left[ O_{12} \begin{pmatrix} -\frac{\delta m^2}{2} & & \\ & +\frac{\delta m^2}{2} & \\ & & \Delta m^2 \end{pmatrix} O_{12}^T + \begin{pmatrix} A/2 & & \\ & -A/2 & \\ & & -A/2 \end{pmatrix} \right]$$

$$= \frac{A}{4E}\mathbb{1} + \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{12}\,\delta m^2 & \sin 2\theta_{12}\,\delta m^2 & 0 \\ \sin 2\theta_{12}\,\delta m^2 & \cos 2\theta_{12}\,\delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix}$$

→ In the primed basis, for $s_{13} \to 0$, the $(e,\mu)$ flavors evolve separately from the $(\tau)$ one

→ $\tilde{S}^{\ell}_{\tau e} = \lim_{s_{13}\to 0} \tilde{S}'_{\tau e} = 0$ (no $\nu_\tau' \to \nu_e'$ transitions)

→ $\tilde{S}'_{\tau e} = O(s_{13}) = O_1$ at least

Instead, $\tilde{S}^{\ell}_{\mu e}$ is nonzero. From the $2\nu$ case (already worked out) we get:

$$\tilde{S}^{\ell}_{\mu e} = e^{-i\frac{A}{4E}x} \cdot \left[ -i \sin 2\tilde{\theta}_{12} \sin\left(\frac{\delta \tilde{m}^2 x}{4E}\right)\right]$$

with $\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} / \sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}$

$\delta\tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$

→ $\tilde{S}^{e}_{\mu e} = O(\delta m^2) = O_1$

Limit $\delta m^2 \to 0$ in primed basis

$$\tilde{H}^h = \lim_{\delta m^2 \to 0} \tilde{H}' = \frac{1}{2E}\left( O_{13} \begin{bmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix} \right)$$

$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E}\right)\mathbb{1} + \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{13}\Delta m^2 & 0 & \sin 2\theta_{13}\Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13}\Delta m^2 & 0 & \cos 2\theta_{13}\Delta m^2 - A \end{bmatrix}$$

→ In the primed basis, for $\delta m^2 \to 0$, the $(e,\tau)$ flavors evolve separately from the $(\mu)$ one

→ $\tilde{S}^h_{e\mu} = \lim_{\delta m^2 \to 0} \tilde{S}'_{e\mu} = 0$ (no $\nu_e' \to \nu_\mu'$ transitions)

→ $\tilde{S}'_{e\mu} = O(\delta m^2) = O_1$ at least.

Instead, $\tilde{S}^h_{e\tau}$ is nonzero. From the $2\nu$ case (already worked out) we get:

$$\tilde{S}^h_{e\tau} = e^{-i\frac{A}{4E}x}\, e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\tilde{\theta}_{13}\,\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\right]$$

with $\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} / \sqrt{(\cos 2\theta_{13} - A/\Delta m^2) + \sin^2 2\theta_{13}}$

$\Delta\tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\hat{\theta}_{13}$

→ $S^h_{e\tau} = O(s_{13}) = O_1$

- Putting all together, at $O_1$ we have that:

$$\tilde{S}'_{e\mu} = O(\delta m^2) \simeq \tilde{S}^{\ell}_{e\mu} = e^{-i\frac{A}{4E}x}\left[-i\sin 2\hat{\theta}_{12}\,\sin\left(\frac{\delta\hat{m}^2 x}{4E}\right)\right]$$

$$\tilde{S}'_{e\tau} = O(s_{13}) \simeq \tilde{S}^{h}_{e\tau} = e^{-i\frac{A}{4E}x}\, e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\tilde{\theta}_{13}\,\sin\left(\frac{\Delta\hat{m}^2 x}{4E}\right)\right]$$

- The overall common phase $e^{-i\frac{A}{4E}x}$ can be eliminated and we have that:

$$\tilde{S}'_{e\mu} = \left[-i\sin 2\hat{\theta}_{12}\,\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

$$\tilde{S}'_{e\tau} = e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\tilde{\theta}_{13}\,\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

- We have now everything we need to calculate $P_{e\mu}$ as a quadratic form in $\tilde{S}'_{e\mu}$ and $\tilde{S}'_{e\tau}$.

 Further tricks involve a proper organization of terms and an expansion in the small parameter $\dfrac{\delta m^2}{A} = \dfrac{\delta m^2}{2\sqrt{2}\,G_F N_e E}$ .

 This is called sometimes "high-energy approximation", and is valid for $E \gtrsim 1\,\mathrm{GeV}$ at typical crust-mantle densities $N_e$-

- More precisely, so far we have:

$$A_{e\mu} = \sin 2\tilde\theta_{12} \sin 2\tilde\theta_{13} \sin 2\theta_{23} \cdot \sin\left(\frac{\Delta\tilde m^2 x}{4E}\right) \sin\left(\frac{\delta\tilde m^2 x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} = \sin 2\tilde\theta_{12} \sin 2\tilde\theta_{13} \sin 2\theta_{23} \cdot \sin\left(\frac{\Delta\tilde m^2 x}{4E}\right) \sin\left(\frac{\delta\tilde m^2 x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} = c_{23}^2 \sin 2\tilde\theta_{12} \sin^2\left(\frac{\delta\tilde m^2 x}{4E}\right) + s_{23}^2 \sin^2 2\tilde\theta_{12} \sin^2\left(\frac{\Delta\tilde m^2 x}{4E}\right)$$

and $P_{e\mu} = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$

The high-energy expansion in $\frac{\delta m^2}{A}$ will allow us to express $\delta\tilde m^2$, $\Delta\tilde m^2$, $\tilde\theta_{12}$, and $\tilde\theta_{13}$, in terms of their vacuum values $\delta m^2$, $\Delta m^2$, $\theta_{12}$, $\theta_{13}$ (together with an expansion in the small parameter $s_{13}$)

- For $\delta m^2/A \ll 1$ :

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} = \frac{\sin 2\theta_{12}}{\sqrt{\cos^2 2\theta_{12} - \frac{2A}{\delta m^2}\cos 2\theta_{12} + \left(\frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}}$$

$$= \frac{\sin 2\theta_{12}}{\sqrt{\left(\frac{A}{\delta m^2}\right)^2\left(1 - 2\frac{\delta m^2}{A}\cos 2\theta_{12} + \cdots\right)}} \simeq \frac{\sin 2\theta_{12}}{\frac{|A|}{\delta m^2}\left(1 - \frac{\delta m^2}{A}\cos 2\theta_{12}\right)} \simeq \sin 2\theta_{12}\,\frac{\delta m^2}{|A|} + O_2$$

$$\frac{\delta m^2}{\delta \tilde{m}^2} = \frac{\sin 2\tilde{\theta}_{12}}{\sin 2\theta_{12}} = \frac{\delta m^2}{|A|} + O_2 \quad \longrightarrow \quad \delta\tilde{m}^2 = |A| + O_2$$

$$\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right) \simeq \sin\left(\frac{|A| x}{4E}\right) + O_2$$

- For $s_{13} \ll 1$ :

$$\sin 2\tilde{\theta}_{13} = \frac{\sin 2\theta_{13}}{\sqrt{\left(\cos 2\theta_{13} - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta_{13}}} \simeq \frac{\sin 2\theta_{13}}{\sqrt{\left(1 - \frac{A}{\Delta m^2}\right)^2}} + O_2 = \frac{\sin 2\theta_{13}}{\left|1 - \frac{A}{\Delta m^2}\right|} + O_2$$

$$\sin 2\tilde{\theta}_{13} = \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{13} + O_2$$

$$\Delta\tilde{m}^2 = \Delta m^2 \frac{\sin 2\theta_{13}}{\sin 2\tilde{\theta}_{13}} \simeq \Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right|$$

We have then:

$$A_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23} \cdot \sin\left(\frac{|A|x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23} \sin\left(\frac{|A|x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) + s_{23}^2 \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\frac{|\Delta m^2 - A|}{4E} x\right)$$

The absolute values can be eliminated by inspection of all relevant $\pm$ cases; e.g., by changing sign of $(\Delta m^2 - A)$: $B_{e\mu}$, $A_{e\mu}$, and $C_{e\mu}$ do not change. By changing the sign of $\Delta m^2$: only $A_{e\mu}$ changes sign. Etc...

Elimination of $|\,,|$ gives:

$$A_{e\mu} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right) \frac{\Delta m^2}{A-\Delta m^2} \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E}x\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right) \frac{\Delta m^2}{A-\Delta m^2} \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E}x\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) + s_{23}^2 \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\frac{\Delta m^2 - A}{4E}x\right)$$

Terms in $P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$ can be finally organized as:

$$P(\nu_e \to \nu_\mu) = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z$$

with

$$\begin{cases} X = \sin^2\theta_{23} \left(\frac{\Delta m^2}{A-\Delta m^2}\right)^2 \sin^2\left(\frac{A-\Delta m^2}{4E}x\right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A-\Delta m^2}\right) \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E}x\right) \\ Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) \end{cases}$$

as desired

- Another way of organizing the terms is:

$$P_{e\mu} = x^2 f^2 + 2xy\, fg \cos(\Delta - \delta) + y^2 g^2$$

with
$$\begin{cases} x = \sin\theta_{23} \sin 2\theta_{13} \\ y = \dfrac{\delta m^2}{\Delta m^2} \cos\theta_{23} \sin 2\theta_{12} \\ f = \sin\left(\dfrac{\Delta m^2 - A}{4E} x\right) \dfrac{\Delta m^2}{\Delta m^2 - A} \\ g = \sin\left(\dfrac{Ax}{4E}\right) \cdot \dfrac{\Delta m^2}{A} \end{cases}$$

- In all cases: for antineutrinos, $A \to -A$, $\delta \to -\delta$

  for inverse hierarchy, $\Delta m^2 \to -\Delta m^2$

**Exercise:** Take the 2nd-order formula for $P_{e\mu}$ in matter, and verify that you get the 2nd-order formula for $P_{e\mu}$ in vacuum (already derived) for $A \to 0$.

The results of this exercise are surprising. In fact:

$$P_{e\mu}^{vac} = \lim_{A \to 0} P_{e\mu}^{matter}$$

is, in our case, a forbidden limit, since we have assumed $A \gg \delta m^2 > 0$ in our derivation!

This means that $P_{e\mu}^{matter}$ is a "lucky" formula, whose range of validity is somewhat larger than expected.

# Exact calculation of $P_{\alpha\beta}$ for $\delta m^2 = 0$ in constant matter

This is the "matter-version" of the one-mass-scale limit, previously examined in vacuum.
In this case, the hamiltonian in the primed basis is:

$$\tilde{H}'(\delta m^2 = 0) = \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{13}\,\Delta m^2 & 0 & \sin 2\theta_{13}\,\Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13}\,\Delta m^2 & 0 & \cos 2\theta_{13}\,\Delta m^2 - A \end{bmatrix} \quad \cancel{(+\,\text{const}\cdot\mathbb{1})}$$

and has thus a $(2\nu)\oplus(1\nu)$ block form, which can be easily diagonalized using previous results:

$$\hat{H}' = \tilde{U}\,\frac{\hat{\mathcal{M}}^2}{2E}\,\tilde{U}^T \qquad \text{with} \qquad \hat{U} = \begin{pmatrix} \cos\tilde{\theta}_{13} & 0 & \sin\hat{\theta}_{13} \\ 0 & 1 & 0 \\ -\sin\tilde{\theta}_{13} & 0 & \cos\tilde{\theta}_{13} \end{pmatrix}$$

$$\sin 2\hat{\theta}_{13} = \sin 2\theta_{13} \Big/ \sqrt{\left(\cos\theta_{13} - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta_{13}}$$

$$\tilde{\mathcal{M}}^2 = \mathrm{diag}\left(-\frac{\Delta\tilde{m}^2}{2},\ \frac{-\Delta m^2 - A}{2},\ +\frac{\Delta\tilde{m}^2}{2}\right) \qquad \text{with} \quad \Delta\tilde{m}^2 \sin 2\hat{\theta}_{13} = \Delta m^2 \sin 2\theta_{13}$$

For later purposes, note that:

$$\begin{cases} \tilde{m}_3^2 - \tilde{m}_1^2 = \Delta\tilde{m}^2 \\ \tilde{m}_3^2 - \tilde{m}_2^2 = (\Delta m^2 + \Delta\tilde{m}^2 + A)/2 \\ \tilde{m}_2^2 - \tilde{m}_1^2 = (-\Delta m^2 - A + \Delta\tilde{m}^2)/2 \end{cases}$$

Since $\tilde{H}'$ is a $(2\nu)\oplus(1\nu)$ block form, also $\tilde{S}' = e^{-i\hat{H}x}$ is a block matrix:

$$\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & 0 & \tilde{S}'_{e\tau} \\ 0 & \tilde{S}'_{\mu\mu} & 0 \\ \tilde{S}'_{ee} & 0 & \tilde{S}'_{\tau\tau} \end{pmatrix}$$

where $\tilde{S}'_{\mu\mu}$ is trivially given by $e^{-i\tilde{H}_{\mu\mu}x}$ :

$$\tilde{S}'_{\mu\mu} = e^{\frac{i}{4E}(\Delta m^2+A)x} = \cos\left(\frac{\Delta m^2+A}{4E}x\right) + i\sin\left(\frac{\Delta m^2+A}{4E}x\right)$$

and the $\begin{pmatrix} ee & e\tau \\ \tau e & \tau\tau \end{pmatrix}$ part already solved in a previous exercise:

$$\tilde{S}'_{(e,\tau)} = \cos\left(\frac{\widetilde{\Delta m}^2}{4E}x\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right)\begin{pmatrix} -\cos 2\hat{\theta}_{13} & \sin 2\hat{\theta}_{13} \\ \sin 2\hat{\theta}_{13} & \cos 2\hat{\theta}_{13} \end{pmatrix} \quad \leftarrow \begin{pmatrix} ee & e\tau \\ \tau e & \tau\tau \end{pmatrix}$$

We also remind that the components of $\tilde{S}$ in the physical flavor basis are given by (for $\tilde{S}'_{e\mu} = \tilde{S}'_{\mu e} = 0 = \tilde{S}'_{\mu\tau} = \tilde{S}'_{\tau\mu}$) :

$$\tilde{S}_{\alpha\beta} = \begin{pmatrix} \tilde{S}'_{ee} & s_{23}\tilde{S}'_{e\tau}e^{-i\delta} & c_{23}\tilde{S}'_{e\tau}e^{-i\delta} \\ s_{23}\tilde{S}'_{\tau e}e^{i\delta} & c_{23}^2\tilde{S}'_{\mu\mu} + s_{23}^2\tilde{S}'_{\tau\tau} & -s_{23}c_{23}\tilde{S}'_{\mu\mu} + s_{23}c_{23}\tilde{S}'_{\tau\tau} \\ c_{23}\tilde{S}'_{\tau e}e^{i\delta} & -s_{23}c_{23}\tilde{S}'_{\mu\mu} + s_{23}c_{23}\tilde{S}'_{\tau\tau} & s_{13}^2\tilde{S}'_{\mu\mu} + c_{23}^2\tilde{S}'_{\tau\tau} \end{pmatrix}$$

Note that, since $\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha}$ (constant matter), and since $\delta$ disappears in $|\tilde{S}_{\alpha\beta}|^2$, all $P_{\alpha\beta}$'s will be CP-conserving and symmetric ($P_{\alpha\beta} = P_{\alpha\beta}$).

Moreover, we remind that only $P_{e\mu}$ and $P_{\mu\tau}$ are needed to calculate all the others:

$$P_{e\mu} = |\tilde{S}_{e\mu}|^2 = s_{23}^2 \, |\tilde{S}'_{e\tau}|^2$$

$$P_{\mu\tau} = |\tilde{S}_{\mu\tau}|^2 = s_{23}^2 c_{23}^2 \, |\tilde{S}'_{\mu\mu} - \tilde{S}'_{\tau\tau}|^2$$

together with the "symmetrical"

$$P'_{e\mu} = P_{e\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} = c_{23}^2 \, |\tilde{S}'_{e\tau}|^2 = P_{e\mu} \cdot \frac{c_{23}^2}{s_{23}^2}$$

$$P'_{\mu\tau} = P_{\mu\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} = s_{23}^2 c_{23}^2 \, |\tilde{S}'_{\mu\mu} - \tilde{S}'_{\tau\tau}|^2 = P_{\mu\tau}$$

Explicitly, once we get $P_{e\mu}$ and $P_{\mu\tau}$, the other relevant probabilities are given by:

$$P_{ee} = 1 - P_{e\mu} - P'_{e\mu} = 1 - P_{e\mu}\left(1 + \frac{c_{23}^2}{s_{23}^2}\right) = 1 - \frac{P_{e\mu}}{s_{23}^2}$$

$$P_{e\tau} = P'_{e\mu} = P_{e\mu}\,\frac{c_{23}^2}{s_{23}^2}$$

$$P_{\mu\mu} = 1 - P_{e\mu} - P'_{\mu\tau} = 1 - P_{e\mu} - P_{\mu\tau}$$

$$P_{\tau\tau} = 1 - P'_{e\mu} - P_{\mu\tau} = 1 - P_{e\mu}\,\frac{c_{23}^2}{s_{23}^2} - P_{\mu\tau}$$

$$(P_{\alpha\beta} = P_{\beta\alpha})$$

So we need $P_{e\mu}$ and $P_{\mu\tau}$ now, i.e., $|\tilde{S}'_{e\tau}|^2$ and $|\tilde{S}'_{\mu\mu} - \tilde{S}'_{\tau\tau}|^2$.

$$\tilde{S}'_{e\tau} = -i\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\sin 2\tilde{\theta}_{13}$$

$$|\tilde{S}'_{e\tau}|^2 = \sin^2 2\tilde{\theta}_{13}\,\sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) = 4\,\tilde{s}^2_{13}\tilde{c}^2_{13}\,\sin^2\left(\frac{\tilde{m}^2_3-\tilde{m}^2_1}{4E}x\right)$$

$$P_{e\mu} = s^2_{23}\,|\tilde{S}'_{e\tau}|^2 = 4\,\tilde{s}^2_{13}\tilde{c}^2_{13}\,s^2_{23}\,\sin^2\left(\frac{\tilde{m}^2_3-\tilde{m}^2_1}{4E}x\right)$$

---

$$\tilde{S}'_{\mu\mu} = \cos\left(\frac{\Delta m^2+A}{4E}x\right) + i\sin\left(\frac{\Delta m^2+A}{4E}x\right)$$

$$\tilde{S}'_{\tau\tau} = \cos\left(\frac{\Delta\tilde{m}^2}{4E}x\right) - i\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}x\right)\cdot\cos 2\tilde{\theta}_{13}$$

$$S'_{\mu\mu} - S'_{\tau\tau} = \left[\cos\left(\frac{\Delta m^2+A}{4E}x\right) - \cos\left(\frac{\Delta\tilde{m}^2}{4E}x\right)\right] + i\left[\sin\left(\frac{\Delta m^2+A}{4E}x\right) + \sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\cos 2\tilde{\theta}_{13}\right]$$

$$|S'_{\mu\mu} - S'_{\tau\tau}|^2 = \cos^2\left(\frac{\Delta m^2+A}{4E}x\right) + \cos^2\left(\frac{\Delta\tilde{m}^2}{4E}x\right) - 2\cos\left(\frac{\Delta m^2+A}{4E}x\right)\cos\left(\frac{\Delta\tilde{m}^2}{4E}x\right)$$
$$+\sin^2\left(\frac{\Delta m^2+A}{4E}x\right) + \sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) + 2\sin\left(\frac{\Delta m^2+A}{4E}x\right)\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\cos 2\tilde{\theta}_{13} =$$

$\times\cos^2 2\tilde{\theta}_{13}$

next →

$$= 2 - \sin^2 2\tilde{\theta}_{13} \sin^2\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right) - 2\left[\cos\left(\frac{\Delta m^2+A}{4E}x\right)\cos\left(\frac{\widetilde{\Delta m}^2}{4E}x\right) - \sin\left(\frac{\Delta m^2+A}{4E}x\right)\cos\left(\frac{\widetilde{\Delta m}^2}{4E}x\right)\left(\tilde{c}_{13}^2 - \tilde{s}_{13}^2\right)\right]$$

$$= -4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right) - 2\left[-1 + \tilde{c}_{13}^2\left(\cos\left(\frac{\Delta m^2+A}{4E}x\right)\cos\left(\frac{\widetilde{\Delta m}^2}{4E}x\right) - \sin\left(\frac{\Delta m^2+A}{4E}x\right)\sin\left(\frac{\widetilde{\Delta m}^2}{4E}x\right)\right)\right.$$
$$\left. + \tilde{s}_{13}^2\left(\cos\left(\frac{\Delta m^2+A}{4E}x\right)\cos\left(\frac{\widetilde{\Delta m}^2}{4E}x\right) + \sin\left(\frac{\Delta m^2+A}{4E}x\right)\sin\left(\frac{\widetilde{\Delta m}^2}{4E}x\right)\right)\right]$$

$$= -4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right) - 2\left[-1 + \tilde{c}_{13}^2 \cos\left(\frac{\Delta m^2+A+\widetilde{\Delta m}^2}{4E}x\right) + \tilde{s}_{13}^2 \cos\left(\frac{\Delta m^2+A-\widetilde{\Delta m}^2}{4E}x\right)\right]$$

$$= -4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right) - 2\tilde{c}_{13}^2\left[\cos\left(\frac{\Delta m^2+A+\widetilde{\Delta m}^2}{4E}x\right) - 1\right] - 2\tilde{s}_{13}^2\left[\cos\left(\frac{\Delta m^2+A-\widetilde{\Delta m}^2}{4E}x\right) - 1\right]$$

$$= -4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\widetilde{\Delta m}^2 x}{4E}\right) + 4\tilde{c}_{13}^2 \sin^2\left(\frac{\Delta m^2+A+\widetilde{\Delta m}^2}{8E}\right) + 4\tilde{s}_{13}^2 \sin^2\left(\frac{\Delta m^2+A-\Delta m^2}{8E}x\right)$$

$$= -4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right) + 4\tilde{c}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_2^2}{4E}x\right) + 4\tilde{s}_{13}^2 \sin^2\left(\frac{\tilde{m}_2^2-\tilde{m}_1^2}{4E}x\right)$$

and $P_{\mu\tau}$ is obtained by multiplying with $s_{23}^2 c_{23}^2$.

$$P_{e\mu} = 4\tilde{s}_{13}^2\tilde{c}_{13}^2\, s_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right)$$

$$\begin{aligned} P_{\mu\tau} = &-4\tilde{s}_{13}^2\tilde{c}_{13}^2\, s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right) \\ &+4\tilde{s}_{13}^2\, s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_2^2-\tilde{m}_1^2}{4E}x\right) \\ &+4\tilde{c}_{13}^2\, s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_2^2}{4E}x\right) \end{aligned}$$

$$P_{ee} = 1 - \frac{P_{e\mu}}{s_{23}^2} = 1 - 4\tilde{s}_{13}^2\tilde{c}_{13}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right)$$

$$P_{e\tau} = P_{e\mu}\frac{c_{23}^2}{s_{23}^2} = 4\tilde{s}_{13}^2\tilde{c}_{13}^2\, c_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right)$$

$$\begin{aligned} P_{\mu\mu} = 1 - P_{e\mu} - P_{\mu\tau} = 1 &- 4\tilde{s}_{13}^2\tilde{c}_{13}^2\, s_{23}^4 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_1^2}{4E}x\right) \\ &- 4\tilde{s}_{13}^2\, s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_2^2-\tilde{m}_1^2}{4E}x\right) \\ &- 4\tilde{c}_{13}^2\, s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2-\tilde{m}_2^2}{4E}x\right) \end{aligned}$$

$$P_{\tau\tau} = 1 - P_{e\mu}\frac{c_{23}^2}{s_{23}^2} - P_{\mu\tau} = 1 - 4\tilde{s}_{13}^2\tilde{c}_{13}^2 c_{23}^4 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_1^2}{4E}x\right)$$
$$- 4\tilde{s}_{13}^2 s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_2^2 - \tilde{m}_1^2}{4E}x\right)$$
$$- 4\tilde{c}_{13}^2 s_{23}^2 c_{23}^2 \sin^2\left(\frac{\tilde{m}_3^2 - \tilde{m}_2^2}{4E}x\right)$$

$$P_{\mu e} = P_{e\mu}$$
$$P_{\tau e} = P_{e\tau}$$
$$P_{\tau\mu} = P_{\mu\tau}$$

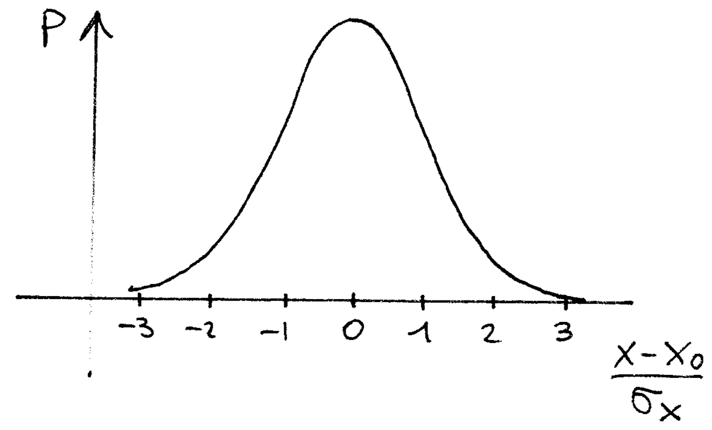
# Elements of statistics

- **Error distribution for one variable**

Here we deal only with gaussian errors. Also Poisson fluctuations ($\propto \sqrt{N}$) are assumed to be nearly gaussian. Distribution for 1 variable $x$:

$$P(x, x_0) = \frac{1}{\sqrt{2\pi} \cdot \sigma_x} \, e^{-\frac{1}{2}\left(\frac{x - x_0}{\sigma_x}\right)^2}$$

corresponding to quote $x = x_0 \pm \sigma_x$



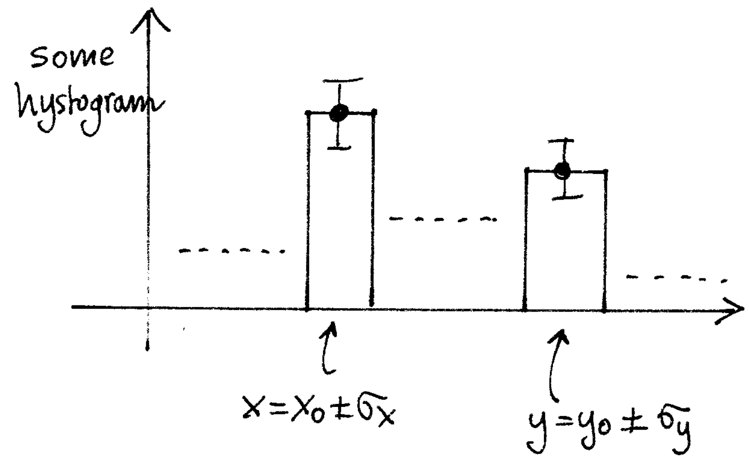

Area within: $\pm 1\sigma$ = 68.27%

$\pm 2\sigma$ = 95.45%

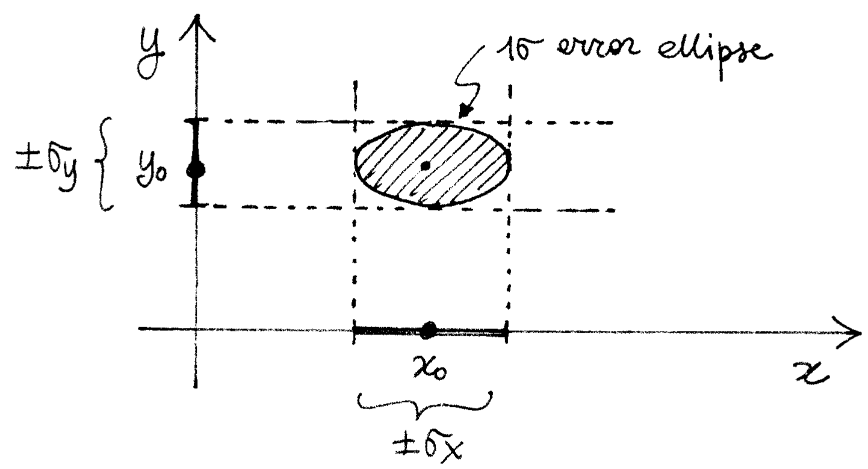
$\pm 3\sigma$ = 99.73%

# Error distribution for 2 variables (uncorrelated case)

Consider two quantities x and y with errors which have no relation with each other (e.g., statistical errors of two bins) :




Plot of $x$ versus $y$ :




1σ Ellipse equation:

$$1 = \left(\frac{x - x_0}{\sigma_x}\right)^2 + \left(\frac{y - y_0}{\sigma_y}\right)^2$$

Define $\Delta\chi^2 = \left(\frac{x - x_0}{\sigma_x}\right)^2 + \left(\frac{y - y_0}{\sigma_y}\right)^2$

(it is zero at $(x_0, y_0)$)

$\rightarrow$ $\Delta\chi^2 = 1$ gives the 1σ error ellipse

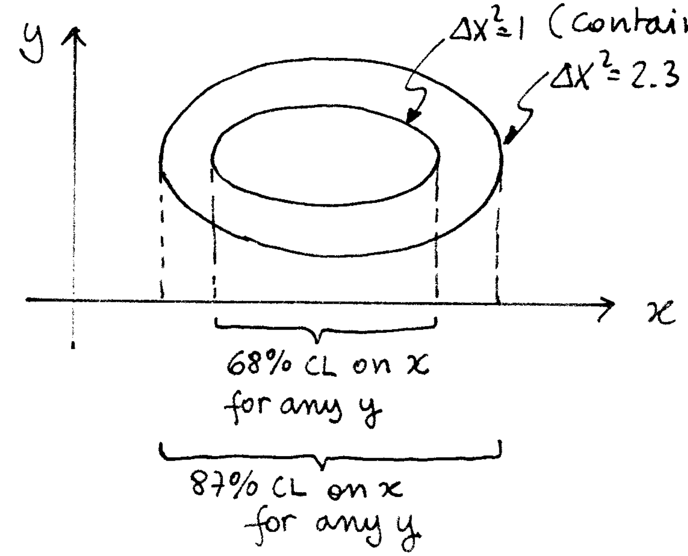
However, probability content of $1\sigma$ error ellipse is <u>less</u> than 68.27 %, and is equal to 39.35% (indeed, must be less than $(68.27\%)^2$ ! )

68 % → probability of $x \in [x_0 - \sigma_x, x_0 + \sigma_x]$ independently on $y$

68 % → " " $y \in [y_0 - \sigma_y, y_0 + \sigma_y]$ " " $x$

39 % → joint probability of $(x,y)$ being inside the $1\sigma$ error ellipse

If you really want an error ellipse containing 68% probability, then must use $\Delta\chi^2 = 2.3$. Its projections define 87 % C.L. for each variable. However, this is not called a "$1\sigma$" ellipse ! (Don't be confused ...)




<u>Therefore</u> : always quote number of variables to which your C.L. refers (one, two, ...)

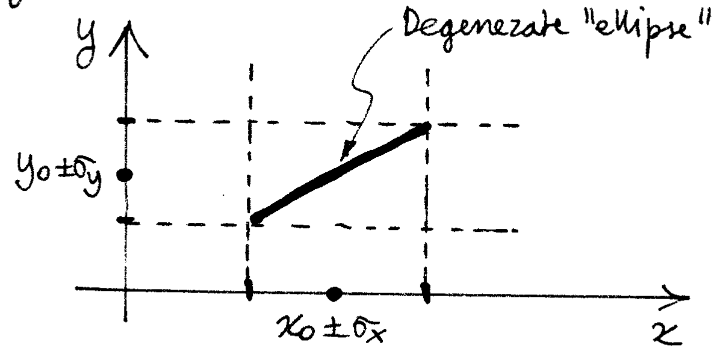
# Error distribution for 2 variables (fully correlated case)

Consider two variables $x$ and $y$ with errors which are in one-to-one correspondence, e.g., two bins affected by a common normalization error:

$$\text{if} \quad x \to x+\sigma_x \quad \text{then} \quad y \to y+\sigma_y$$
$$\text{if} \quad x \to x-\sigma_x \quad \text{then} \quad y \to y-\sigma_y$$

(cannot have $x+\sigma_x$ <u>and</u> $y-\sigma_y$)

Then the "error ellipse" is degenerate:

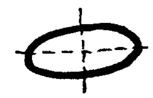



Analogously for full "anticorrelation"
(e.g.: two bins whose sum is constant; then $x \to x_0+\sigma_x$ implies that $y \to y_0-\sigma_y$, and the "degenerate ellipse" has a negative slope)

# Recap of limiting $1\sigma$ error ellipses

- Equation for no correlation: $(x-x_0, y-y_0)\begin{pmatrix}\sigma_x^2 & 0\\ 0 & \sigma_y^2\end{pmatrix}^{-1}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = 1$

  det $\neq 0$



- Equation for full correlation: $(x-x_0, y-y_0)\begin{pmatrix}\sigma_x^2 & \sigma_x\sigma_y\\ \sigma_x\sigma_y & \sigma_y^2\end{pmatrix}^{-1}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = 1$

  det $= 0$, degenerate

- Equation for full anticorrelation: $(x-x_0, y-y_0)\begin{pmatrix}\sigma_x^2 & -\sigma_x\sigma_y\\ -\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}^{-1}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = 1$

  det $= 0$, degenerate

$$(x-x_0, y-y_0)\begin{pmatrix}\text{squared}\\ \text{error}\\ \text{matrix}\end{pmatrix}^{-1}\begin{pmatrix}x-x_0\\ y-y_0\end{pmatrix} = 1$$

# Generalization of $1\sigma$ error ellipses

Suppose you have for $x$ and $y$ two sources of errors:

statistical $(s_x, s_y)$ with no correlation

systematic $(c_x, c_y)$ with full correlation

$$\begin{cases} x = x_0 \pm s_x \pm c_x \\ y = y_0 \pm s_y \pm c_y \end{cases}$$

Then error matrices sum up in quadrature:

$$\sigma^2 = \begin{bmatrix} s_x^2 & 0 \\ 0 & s_y^2 \end{bmatrix} + \begin{bmatrix} c_x^2 & c_x c_y \\ c_x c_y & c_y^2 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \sigma_x^2 & \rho\,\sigma_x\sigma_y \\ \rho\,\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

where:

$$\sigma_x^2 = s_x^2 + c_x^2$$

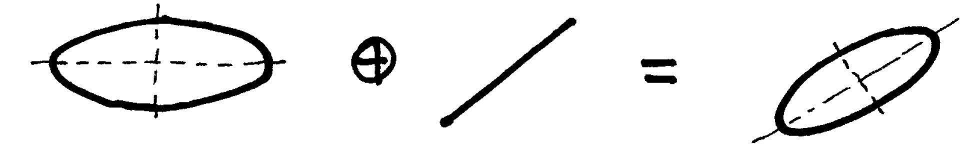
$$\sigma_y^2 = s_y^2 + c_y^2$$

$$\rho = \frac{c_x c_y}{\sigma_x \sigma_y} = \begin{cases} 0 & \text{for } c_x \text{ or } c_y = 0 \quad \text{(no correlation)} \\ 1 & \text{for } s_x = s_y = 0 \quad \text{(full correlation)} \end{cases}$$

In general, $0 \le |\rho| \le 1$

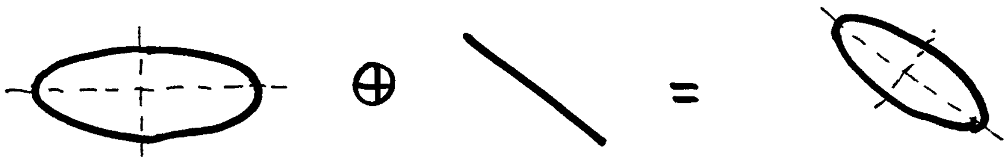
What about the shape of the $1\sigma$ error ellipse ?

$$\Delta\chi^2 = 1 \;\rightarrow\; (x-x_0,\, y-y_0)\begin{pmatrix}\sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}^{-1}\begin{pmatrix}x-x_0 \\ y-y_0\end{pmatrix}$$



$\rho = 0$ $\qquad \rho = 1$ $\qquad 0<\rho<1$

Analogously, for the addition of a fully anticorrelated error source :

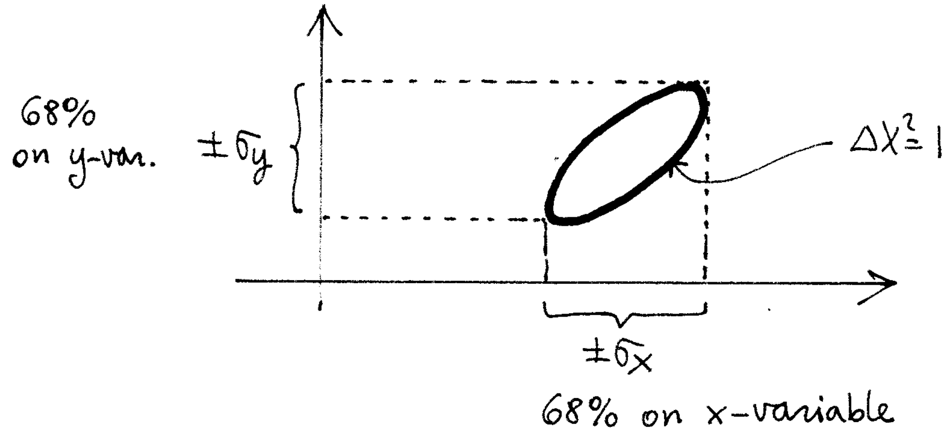


$\rho = 0$ $\qquad \rho = -1$ $\qquad -1<\rho<0$

Explicit equation for any $\rho$ is:

$$\frac{1}{(1-\rho^2)}\left[\left(\frac{x-x_0}{\sigma_x}\right)^2 + \left(\frac{y-y_0}{\sigma_y}\right)^2 - 2\rho\,\frac{(x-x_0)(y-y_0)}{\sigma_x\sigma_y}\right] = 1$$

- Note: projections of $\Delta\chi^2 = 1$ ellipse always give total $1\sigma$ errors:




- Probability content of joint gaussian distribution:

$$P = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\Delta\chi^2}$$

; for $\Delta\chi^2 \leq 1$ is again 39%
(joint CL for 2 variables at $\Delta\chi^2 = 1$)

# Generalization to N variables $\{x_i\}_{1 \leq i \leq N}$

- $\Delta\chi^2 = 1$ error ellipse defined by :

$$\Delta x^T \sigma^{-2} \Delta x = 1$$

Where $\Delta x = \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_N - x_N^0 \end{pmatrix}$ $\quad \sigma^{-2} = (\sigma^2)^{-1}$

$$\sigma^2 = \begin{pmatrix} \sigma_1^2 & \rho_{12}\,\sigma_1\sigma_2 & \rho_{13}\,\sigma_1\sigma_3 & \cdots \\ & \sigma_2^2 & \rho_{23}\,\sigma_2\sigma_3 & \cdots \\ & & \sigma_3^2 & \cdots \\ & & & \ddots \end{pmatrix}$$ (symmetrical)

- Probability distribution (multivariate gaussian):

$$P = \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det\sigma^2}} \cdot e^{-\frac{1}{2}\Delta x^T \sigma^{-2} \Delta x} = \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det\sigma^2}} \cdot e^{-\frac{1}{2}\Delta\chi^2}$$

- Projection of N-dimensional $\Delta\chi^2 = 1$ ellipsoid onto the axis $x_i$ gives the range $x_0^i \pm \sigma_i$ ($\pm 1\sigma$ on $x_i$ variable, i.e, 68% CL on that variable for any value of the others $x_j \neq x_i$). This is true for <u>any</u> N. Variables $\neq x_i$ are said to be "marginalized" or "projected away".

- However, probability of $(x_1, x_2, \dots, x_N)$ being <u>jointly</u> within the ellipsoid, $$P' = \int_{\Delta\chi^2 \leq 1} P \quad ,$$

decreases with N:

| | | |
|---|---|---|
| $N=1$ | 68% | |
| $N=2$ | 39% | $< (68\%)^2$ |
| $N=3$ | 20% | $< (68\%)^3$ |
| $N=4$ | 9% | $< (68\%)^4$ |
| ⋮ | ⋮ | |

How is the matrix $\sigma^2 = \sigma^2_{ij} = \rho_{ij}\,\sigma_i \sigma_j$ $(\rho_{ii} = 1)$ built, in practice?

- Either experimentalists give you $\sigma_i^{exp}$ and $\sigma_j^{exp}$ as well as $\rho_{ij}^{exp}$ ...
  (then, if needed, you evaluate $\sigma_i^{theo}$, $\sigma_j^{theo}$, and $\rho_{ij}^{theo}$)
  $\rightarrow \sigma^2_{ij} = \rho_{ij}^{exp}\,\sigma_i^{exp}\,\sigma_j^{exp} + \rho_{ij}^{theo}\,\sigma_i^{theo}\,\sigma_j^{theo}$

- ... Or all possible error sources are broken down into, say, $\rho=0$ and $\rho=1$ components ($\rho=-1$ can be tuned into $\rho=1$ by changing sign to some $\sigma$'s).

Then
$$\sigma^2_{ij} = \underbrace{\begin{pmatrix} \sigma_1^{2\,\prime} & & \\ & \sigma_2^{2\,\prime} & \\ & & \ddots \end{pmatrix}}_{\text{rank N matrix of fully uncorrelated components}} + \underbrace{\begin{pmatrix} \sigma_1^{2\,\prime\prime} & \sigma_1''\sigma_2'' & \sigma_1''\sigma_3'' & \vdots \\ & \sigma_2^{2\,\prime\prime} & \sigma_2''\sigma_3'' & \vdots \\ & & \ddots & \end{pmatrix}}_{\text{rank 1 matrix of fully correlated components}} + \begin{pmatrix} \cdots \end{pmatrix} + \cdots$$

# Fitting data with a model

- Build $\chi^2 = \Delta x^T (\sigma^2)^{-1} \Delta x$ , $\Delta x = \begin{pmatrix} x_1^{theo}(\vec{p}) - x_1^{exp} \\ x_2^{theo}(\vec{p}) - x_2^{exp} \\ \vdots \\ x_N^{theo}(\vec{p}) - x_N^{exp} \end{pmatrix}$

  $\vec{p}$ = parameter space of the model (dimension $N_p$)

- Find $\chi^2_{min} = \min_{\vec{p}} \chi^2(\vec{p})$ at $\vec{p} = \vec{p}_0$

- Check that $\chi^2_{min} \sim \underbrace{N - N_p}_{\text{degrees of freedom for test of hypothesis}} \pm \sqrt{2(N-N_p)}$ ← see, e.g. Particle Data Book

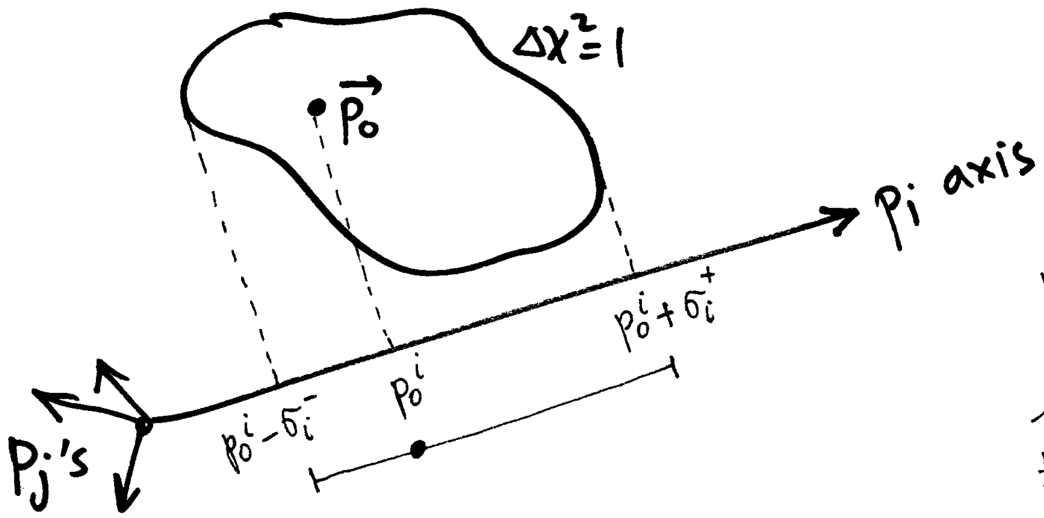
- If previous check not satisfied, model is either wrong ($\chi^2_{min}$ too high) or "too good" and suspect ($\chi^2_{min}$ too low) → can find tables to quantify statement

- If previous checks satisfied, try to estimate parameters around best-fit point $\vec{p}_0$ (parameter estimation)

# Parameter estimation

Suppose you want $\pm 1\sigma$ ranges for each parameter $p_1, p_2, p_3, \dots$ independently on the others (= "marginalizing" the others).

Then:
- Build $\Delta\chi^2 = \chi^2(\vec{p}) - \chi^2_{min}$ ← $N_p$ dimensional manifold
- Project $\Delta\chi^2$ onto axis $p_i$ ← Get $p_i^0 {}^{+\sigma_p^+}_{-\sigma_p^-}$ (asymmetric errors in general)




What does "projection" mean in practice? With a little thought, $1\sigma$ range on $p_i$ equivalent to take $\Delta\chi^2_i = 1$, where

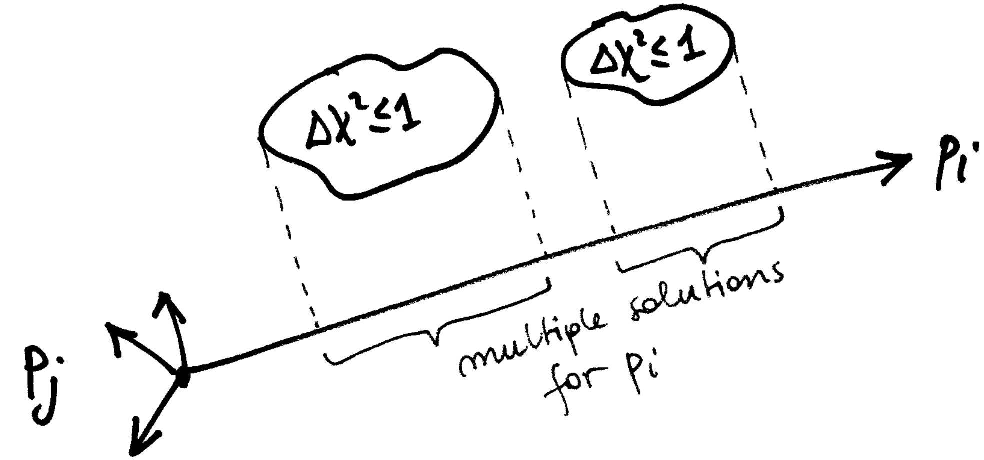
$$\Delta\chi^2_i = \min_{p_j \neq p_i} \left(\chi^2(\vec{p}) - \chi^2_{min}\right)$$

(reduced $\Delta\chi^2$ for variable $p_i$, the others being marginalized)

- This procedure ( $\Delta\chi^2 = \text{const}$ projections) can be justified, as far as the allowed manifold is a simply connected volume.
  (The basic idea is that, through mapping, this volume can be transformed into an ellipsoid → gaussian theorems etc.)

- For disconnected regions, there is no real statistical motivation to use " $\Delta\chi^2 = \text{const}$ " recipe → but no real or "consensus" alternative !

- Apart from isolated or "ad hoc" attempts, basically everybody uses $\Delta\chi^2 = \text{const}$ also in case of multiple domains of $\vec{p}$, ... with some cautionary remarks.




... keep both solutions and wait for experiments to solve the ambiguity ("degeneracy of solutions")

- Sometimes, one is interested not only in $\pm 1\sigma, \pm 2\sigma, \pm 3\sigma, \ldots$ limits on each variable separately ($\equiv \Delta\chi^2 = 1, 4, 9, \ldots$ projections), but on the <u>joint</u> probability of $\vec{p}$ being in a volume defined by $\Delta\chi^2 = $ const.

Relevant table of $\Delta\chi^2$ level cuts:

| C.L. % | N = 1 | N=2 | N=3 | |
|---|---|---|---|---|
| 68.27 | 1.00 | 2.30 | 3.53 | (from PDB) |
| 90 | 2.71 | 4.61 | 6.25 | |
| 95 | 3.84 | 5.99 | 7.82 | |
| 99 | 6.63 | 9.21 | 11.34 | |
| 99.73 | 9.00 | 11.83 | 14.16 | |

So: 95% C.L. on one variable $p_i$ (the others being unconstrained) means $\Delta\chi^2_i = 3.84 \simeq 4$ (= $2\sigma$ squared) $\leftarrow$ cut on reduced $\Delta\chi^2$

If you have two variables (e.g., $\delta m^2$ and $\theta_{12}$), 95% C.L. jointly means $\Delta\chi^2 = 5.99 \simeq 6$

# Splitting $\chi^2$ into pieces

- The $\chi^2$ is a global quantity; by itself, a large $\Delta\chi^2$ doesn't tell you which observable is being badly fitted $\rightarrow$ Need to "split" $\chi^2$ into pieces and check.

- One possibility is to look at the residuals in the (local) best-fit point $\vec{p}_0$:

$$\chi^2_{min} = \sum_{ij=1}^{N} \left(x_i^{theo}(\vec{p}_0) - x_i^{exp}\right) (\sigma^2)^{-1}_{ij} \left(x_j^{theo}(\vec{p}_0) - x_j^{exp}\right)$$

$$\rightarrow \quad \left\{ \left(x_i^{theo}(\vec{p}_0) - x_i^{exp}\right) \right\}_{i=1,\dots,N} \quad = \text{set of residuals}$$

- It is useful to normalize them to $\sqrt{\sigma^2_{ii}}$ (total error on $x_i$)

$$\rightarrow \quad \left\{ \frac{x_i^{theo}(\vec{p}_0) - x_i^{exp}}{\sqrt{\sigma^2_{ii}}} \right\}_{i=1,\dots,N} = \text{set of pulls}$$

- Then a large pull (say $\gtrsim 3\sigma$) tells you where problems are likely to emerge.

- With the previous definitions, however,

$$\chi^2 \neq \sum (\text{pull})^2$$

since $\sigma^2_{ij}$ is not diagonal in general.

- It is possible to put the $\chi^2$ in a form $\sum (\text{pull})^2$ with a slightly different definition, which also brings some technical advantages.

Previous $\chi^2$ approach : "covariance method"
Alternative $\chi^2$ " : "pull" method

# Covariance approach

- Consider N observables $\{R_n\}_{n=1,\dots,N}$ :

  $\{R_n^{theo}\}$ = theoretical predictions

  $\{R_n^{exp}\}$ = experimental determinations

$$\left(R_n^{expt} - R_n^{theo}\right) \pm \underset{\substack{\uparrow \\ \text{uncorrel.} \\ \text{error}}}{u_n} \pm \underbrace{c_n^1 \pm c_n^2 \pm \dots \pm c_n^K}_{\substack{\text{Set of } K \text{ systematics} \\ \text{(produced by independent sources)}}}$$

with $\rho(u_n, u_m) = 0$ (always uncorrelated)

$\rho(c_n^k, c_m^k) = 1$ (fully correlated for same k-th source)

$\rho(c_n^k, c_m^h) = 0$ ($h \neq k$, uncorrelated from different sources)

- Then: Build $\sigma_{nm}^2 = \delta_{nm} u_n u_m + \sum_{k=1}^{K} c_n^k c_m^k$

  and evaluate $\chi^2_{cov} = \sum_{n,m=1}^{N} \left(R_n^{expt} - R_n^{theo}\right) \underset{\substack{\uparrow \\ \text{covariance matrix}}}{\left[\sigma_{nm}^2\right]^{-1}} \left(R_m^{exp} - R_m^{theo}\right)$

  (just as defined previously)

# Pull approach

- Shift the theoretical predictions linearly in the systematics:

$$R_n^{theor} \rightarrow R_n^{theor} + \sum_{k=1}^{K} \xi_k c_n^k$$

where $\xi_k$ = univariate gaussian random error ( $\langle \xi_k \rangle = 0$ , $\langle \xi_k^2 \rangle = 1$ )

- Minimize over $\xi_k$ the following sum of squared residuals:

$$\chi^2_{pull} = \min_{\{\xi_k\}} \Bigg[ \underbrace{\sum_{n=1}^{N} \left( \frac{R_n^{expt} - \left(R_n^{theor} + \sum_{k=1}^{K} \xi_k c_n^k\right)}{u_n} \right)^2}_{\text{squared residuals}} + \underbrace{\sum_{k=1}^{K} \xi_k^2}_{\text{penalty term}} \Bigg]$$

- Note ; at minimum ( $\xi_k \stackrel{def}{=} \bar{\xi}_k$ ) :

$$\chi^2_{pull} = \sum_{n=1}^{N} \left( \frac{R_n^{expt} - \bar{R}_n^{theo}}{u_n} \right)^2 + \sum_{k=1}^{K} \bar{\xi}_k \qquad \Big\} \text{ "diagonal" form}$$

where $\bar{R}_n^{theo} = R_n^{theo} + \sum_{k=1}^{K} \bar{\xi}_k c_n^k$ ← "shifted" predictions

$$\rightarrow \chi^2_{pull} = \sum_{n=1}^{N} \begin{pmatrix} \text{pull of} \\ \text{observable} \end{pmatrix}_n^2 + \sum_{k=1}^{K} \begin{pmatrix} \text{pull of} \\ \text{systematic} \end{pmatrix}_k^2$$

- It turns out that: $\chi^2_{pull} \equiv \chi^2_{covariance}$ (some algebra needed)

  so the methods are numerically equivalent.

- Looking at $\chi^2_{pull}$ in more detail, the "pull" approach appears to be more convenient from a technical viewpoint, especially for large N.

- In addition, one gets an idea of the size of $\bar{\xi}_k$, namely, of the preferred offset of the k-th systematic error/source from zero.
  The larger the $\bar{\xi}_k$'s, the more the fit tends to "stretch" the systematics to accommodate the data in the theory (or the theory in the data, if you want).